MATH8510
Lecture 1 Notes

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What are the p-adic numbers?

Fix a prime number $p \in \mathbb{Z}$.

Definition. The **field of p-adic numbers** $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic absolute value

$$|\cdot|_p : \mathbb{Q} \to \mathbb{R}_{\geq 0}$$

defined by $|0|_p := 0$ and $|x|_p := p^{-\operatorname{ord}_p(x)}$, where $\operatorname{ord}_p(x)$ is the unique $v \in \mathbb{Z}$ such that $x = p^v \cdot \frac{a}{b}$ with $p \nmid a$ and $p \nmid b$.

Observe from the above definition that $|x|_p \leq 1$ for all $x \in \mathbb{Z}$. In particular, whenever $x \in p^n\mathbb{Z}$, we have that $|x|_p \leq p^{-n}$.

Initial roadmap to the p-adics.

1. First, we construct $\mathbb{Q}_p$.
2. Then, we will show that every nonzero $x \in \mathbb{Q}_p$ can be written uniquely as a series of the form

$$x = \sum_{n=v}^{\infty} \alpha_n p^n$$

where $\alpha_n \in \{0, 1, 2, \ldots, p-1\}$ and $\alpha_v \neq 0$. Note that v may be any element of $\mathbb{Z}$.

Assuming the above series expression for x, we see that $x \in \mathbb{N} \iff v \geq 0$ and the digits d_n eventually terminate. As for the negative integers, first observe that

$$\begin{aligned} \sum_{n=0}^{\infty} p^n &= \lim_{m\to\infty} \sum_{n=0}^{m} p^n = \lim_{m\to\infty} \frac{p^{m+1}-1}{p-1} = -\frac{1}{p-1} \\ \implies \sum_{n=0}^{\infty} (p-1)\, p^n &= -1. \end{aligned}$$

Example (informal)**.** $-1 \in \mathbb{Q}_5$ can be written as

$$-1 = \sum_{n=0}^{\infty} 4 \cdot 5^n$$

or, in "decimal form" we have $-1 = (\ldots 444444)_5$. As for -3, we can write $-1 \cdot 3$ in "decimal form" and multiply to compute:

$$\begin{array}{rr} & \ldots 4444 \\ + & \ldots 0003 \\ \hline & \ldots 4442 \end{array}$$

We can check that $-3 = (\ldots 44442)_5$ by adding 3, and seeing that . A similar geometric series trick gives expansions for all $\frac{a}{b} \in \mathbb{Q}$.

Remark. For the localization $\mathbb{Z}_{(p)}$, the same process of power series expansion works.

In place of $\mathbb{Z}, \mathbb{Q}, p$, consider $\mathbb{C}[x], \mathbb{C}(x)$, and a prime ideal $\mathfrak{p}$. There is a unique $\alpha \in \mathbb{C}$ such that $\mathfrak{p} = (x - \alpha)$ and every nonzero $\frac{p(x)}{q(x)}$ has a unique formal Laurent expansion

$$\frac{p(x)}{q(x)} = \sum_{n=v}^{\infty} c_n (x - \alpha)^n$$

with $c_n \in \mathbb{C}$ and $c_v \neq 0$ (although v may be any integer). This is called the **field of Laurent series**, and is denoted $\mathbb{C}((x - \alpha))$.

In fact, every element of the localization $\mathbb{C}[x]_{\mathfrak{p}}$ may be written in the form $\frac{p(x)}{q(x)}$, where $p(x) \in \mathbb{C}[x]$ and $q(x) \in \mathbb{C}[x] \setminus \mathfrak{p}$. Moreover, saying $c_n \in \mathbb{C} \cong \mathbb{C}[x]/\mathfrak{p}$ is just like saying $\alpha_n \in \{0, 1, 2, \ldots, p-1\}$ " = "$\mathbb{Z}/p\mathbb{Z}$.

Considering the embedding $\mathbb{C}(x) \hookrightarrow \mathbb{C}((x - \alpha))$ is what is meant by "thinking about $\mathbb{C}(x)$ locally near α." This is especially useful if $(x - \alpha)$ is understood to be small.

Considering the embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ is what is meant by "thinking about $\mathbb{Q}$ locally near p." This is especially useful analytically if p is understood to be small.

You might ask why it's important to think of $\mathbb{Q}$ locally near p. The claim is that thinking of $\mathbb{Q}$ *only* as a subfield of $\mathbb{R}$ obscures its "arithmetic content." The following example attempts to illustrate this:

Example. Plotting $x = \frac{998}{999}, y = \frac{999}{1000}$, and 1 on the real line makes it seem like $x \approx y \approx 1$. But under p-adic absolute values, we see big differences:

	$p=2$	$p=3$	$p=5$	$p=37$	$p=499$	all other primes
$\lvert x\rvert_p$	$\frac{1}{2}$	27	1	37	$\frac{1}{499}$	1
$\lvert y\rvert_p$	8	$\frac{1}{27}$	25	$\frac{1}{37}$	1	1
$\lvert y\rvert_p$	1	1	1	1	1	1

The Local-to-Global Principle

To study/prove things about $\mathbb{Q}$, we study **all** of the local fields

$$\mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \ldots, \mathbb{Q}_\infty = \mathbb{R}$$

which we embed $\mathbb{Q}$ into.

Theorem (Artin's Product Formula). *If* $x \in \mathbb{Q}^\times$, *then* $\prod_{p \leq \infty} |x|_p = 1$.