MATH8510
Lecture 11 Notes

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Preparing for Kummer Congruences

Bernoulli Numbers and Zeta Functions

Definition. The **Bernoulli polynomials** $(B_n(x))_{n=0}^{\infty}$ are implicitly defined for all $x \in \mathbb{C}$ by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}$$

Note that the above series converges absolutely for $z \in \mathbb{C}$ satisfying $|z| < 2\pi$, so we can use Taylor's formula to compute the $B_n(x)$ terms:

$$B_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(\frac{ze^{xz}}{e^z - 1} \right) \bigg|_{z=0}$$

We can immediately compute the first few of these polynomials:

$$\begin{aligned} B_0(x) &= 1 \\ B_1(x) &= x - \frac{1}{2} \\ B_2(x) &= x^2 - x + \frac{1}{6} \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \end{aligned}$$

and so on.

Definition. For each $n \geq 0$, the $n^{\mathbf{th}}$ **periodic Bernoulli polynomial** is the function

$$\begin{aligned} P_n &\colon \mathbb{R} \to \mathbb{R} \\ P_n(X) &:= B_n(\lfloor X \rceil). \end{aligned}$$

The $n^{\mathbf{th}}$ **Bernoulli number** is $B_n := B_n(0)$.

We can immediately observe the following properties:

i. $\sum_{n=0}^{\infty}(B_n(1)-B_n(0))\frac{z^n}{n!} = \frac{ze^z}{e^z-1} - \frac{z}{e^z-1} = z = \frac{z^1}{1!}$. So $B_n(1) = B_n(0)$ for all $n \neq 1$. Consequently, each P_n is continuous for $n \neq 1$.

ii. The function $f\colon B_{2\pi}(0) \to \mathbb{C}$ (where $B_{2\pi}(0)$ is the disk of radius 2π in $\mathbb{C}$) given by $f(z) = \frac{z}{2}\left(\frac{e^z+1}{e^z-1}\right)$ satisfies $f(-z) = f(z)$, so f is an even function of z. But

$$\begin{aligned} f(z) &= \frac{z}{2}\left(1 + \frac{2}{e^z-1}\right) = \frac{z}{2} + \sum_{n=0}^{\infty} B_n \cdot \frac{z^n}{n!} \\ &= B_0 + \left(B_1 + \frac{1}{2}\right)z + B_2 \cdot \frac{z^2}{2!} + B_3 \cdot \frac{z^3}{3!} + \cdots \end{aligned}$$

So $B_1 = -\frac{1}{2}$ and $B_n = 0$ for all odd $n > 1$.

Theorem (Kummer's Congruence, 1851)**.** *Fix $p > 2$, $r \in \{1, 2, \ldots, p-2\}$, and define*

$$S_r \coloneqq \{r + m(p-1) \mid m \in \mathbb{Z}_{\geq 0}\}.$$

Then for all $k \in S_r$, $\frac{B_k}{k} \in \mathbb{Z}_{(p)}$, and for all $k, k' \in S_r$,

$$\left(1 - p^{k-1}\right)\frac{B_k}{k} \equiv \left(1 - p^{k'-1}\right)\frac{B_{k'}}{k'} \pmod{p^{n+1}\mathbb{Z}_{(p)}}$$

whenever $k \equiv k' \pmod{p^n\mathbb{Z}}$.

Proof. Since $\mathbb{Z}_{(p)} \subseteq \mathbb{Z}_p$, we have a function $g\colon S_r \to \mathbb{Z}_p$ defined by $g(k) \coloneqq \left(1 - p^{k-1}\right)\frac{B_k}{k}$, and g satisfies

$$|k - k'|_p \leq p^{-n} \implies |g(k) - g(k')|_p \leq p^{-(n+1)}$$

for all $k, k' \in S_r$. That is, g is *uniformly* continuous on S_r, hence extends uniquely to a continuous function $\mathbb{Z}_p \to \mathbb{Z}_p$. ∎

What is g, really?

Exercise. Show that S_r is dense in $\mathbb{Z}_p$.

A Foray into Fourier Series

Definition. Suppose $f : \mathbb{R} \to \mathbb{C}$ is integrable on $[0, 1]$. The **Fourier coefficients** of f are defined by

$$c_n \coloneqq \int_0^1 f(x)e^{-2\pi i n x}\,\mathrm{d}x$$

for all $n \in \mathbb{Z}$. The **Fourier series** for f is defined by

$$S_f(x) \coloneqq \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x}$$

for all $x \in \mathbb{R}$ such that the series converges.

Theorem ($\sim$ 1820, Corollary of a theorem of Dirichlet)**.** *If $f\colon \mathbb{R} \to \mathbb{C}$ is 1-periodic, continuous, and differentiable on $\mathbb{R} \setminus \mathbb{Z}$ with bounded derivative, then $S_f(x)$ converges absolutely uniformly to $f(x)$. In particular, we may say that*

$$f(x) = S_f(x).$$

We will not prove this for the sake of time. We will apply this to P_m with $m \neq 1$.

Proposition. *For $m \geq 2$, P_m is given by this absolutely uniformly convergent series:*

$$P_m(x) \coloneqq -\frac{2 \cdot m!}{(2\pi)^m} \sum_{n=1}^{\infty} \frac{1}{n^m} \cos\left(2\pi n x - \frac{\pi}{2} m\right).$$

Proof. Exercise. ■

Consequently, if $k \in \mathbb{N}$, we have

$$\begin{aligned} B_{2k} = P_{2k}(0) &= -\frac{2(2k)!}{(2\pi)^{2k}} \cdot \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \cos(-\pi k) \\ &= (-1)^{k+1} \cdot \frac{2(2k)!}{(2\pi)^{2k}} \cdot \zeta(2k). \end{aligned}$$

Corollary. *For all $k \in \mathbb{N}$, $\zeta(2k) = \frac{(2\pi)^{2k}}{2(2k)!}(-1)^{k+1} B_{2k}$.* ■

We enumerate the first few terms:

$$(\zeta(2k))_{k=1}^{\infty} = \left(\frac{\pi^2}{6}, \frac{\pi^4}{80}, \frac{\pi^6}{945}, \dots\right)$$

and so on.

Proposition (Some facts about ζ)**.**

i. For all $s \in \mathbb{C}$ with $\Re(s) > 1$, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ can be written as an Euler product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

ii. ζ is holomorphic on its domain, i.e.

$$\zeta'(s) = \lim_{h \to 0} \frac{\zeta(s+h) - \zeta(s)}{h}$$

exists for all $s \in \mathbb{C}$ satisfying $\zeta(s) > 1$.