MATH8510
Lecture 12 Notes

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Important Gadgets from Fourier Analysis

Definition. The space of functions $f\colon \mathbb{R} \to \mathbb{C}$ with finite 1-norm, i.e. all such f satisfying

$$\|f\| := \int_{\mathbb{R}} |f(x)| \,\mathrm{d}x < \infty,$$

is denoted $L^1(\mathbb{R})$.

Definition. The space of **Schwartz-Bruhat functions** $\mathcal{S}(\mathbb{R})$ may be defined as

$$\mathcal{S}(\mathbb{R}) := \left\{ f\colon \mathbb{R} \to \mathbb{C} \mid m^n D^k f \text{ is bounded } \forall n, k \geq 0 \right\}$$

where m and D are the operators defined by $(mf)(x) := 2\pi i \cdot f(x)$, and $D = \frac{\mathrm{d}}{\mathrm{d}x}$.

Proposition. *$\left(L^1(\mathbb{R}), \|\cdot\|\right)$ is a complete normed $\mathbb{C}$-vector space (Banach space), and $\mathcal{S}(\mathbb{R})$ is a dense subspace of $L^1(\mathbb{R})$.*

We will not prove this proposition, as it is usually covered in a first graduate course in real analysis.[1]

The "normalized Gaussians" $g_c\colon \mathbb{R} \to \mathbb{C}$ given by

$$g_c(x) := \sqrt{\frac{c}{\pi}} \cdot e^{-cx^2}$$

with $c > 0$ are important elements of $\mathcal{S}(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} g_c(x) \,\mathrm{d}x = 1.$$

Definition. If $f \in L^1(\mathbb{R})$, its **Fourier transform** $\widehat{f}\colon \mathbb{R} \to \mathbb{C}$ is defined by

$$\widehat{f}(t) := \int_{\mathbb{R}} f(x) e^{-2\pi i t x} \,\mathrm{d}x.$$

[1] One reference which may be of use is Rudin's "Real and Complex Analysis."

We have a few basic facts about the Fourier transform:

(a) The Fourier transform is a $\mathbb{C}$-linear map

$$\widehat{(-)} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$$

where $C_0(\mathbb{R}) = \{g \colon \mathbb{R} \to \mathbb{C} \mid |g(t)| \to 0 \text{ as } |t| \to \infty\}$.

(b) Suppose $f \in L^1(\mathbb{R})$.

i. If $\alpha \in \mathbb{R}^\times$, then

$$\widehat{f(\alpha x + \beta)}(t) = \frac{e^{-2\pi i \beta t}}{|\alpha|} \cdot \widehat{f}\left(\frac{t}{\alpha}\right)$$

ii. If Df exists and $Df \in L^1(\mathbb{R})$, then $\widehat{Df} = m\widehat{f}$, i.e. the following diagram commutes:

$$\begin{array}{ccc} f & \overset{D}{\longmapsto} & Df \\ \widehat{(-)} \big\downarrow & & \big\downarrow \widehat{(-)} \\ \widehat{f} & \underset{m}{\longmapsto} & m\widehat{f} = \widehat{Df} \end{array}$$

when it makes sense.

iii. If $Df, D^2 f, D^3 f, \ldots, D^n f \in L^1(\mathbb{R})$, then induction yields

$$\widehat{D^n f} = m^n \widehat{f}$$

and therefore for some $c \geq 0$,

$$\left|\widehat{f}(t)\right| \leq \frac{c}{|t|^n}$$

In other words, the more smooth that f is, the faster that $\widehat{f}$ decays.

iv. Similarly, if $mf, m^2 f, \ldots, m^n f \in L^1(\mathbb{R})$, induction yields that $\widehat{m^n f} = (-D)^n \widehat{f}$, and thus $\widehat{f}$ is n-differentiable. In other words, the faster that f decays, the smoother that $\widehat{f}$ is.

(c) Restricting the Fourier transform to $\mathcal{S}(\mathbb{R})$ yields an automorphism $\mathcal{S}(\mathbb{R}) \xrightarrow{\widehat{(-)}} \mathcal{S}(\mathbb{R})$ that is L^2-isometric, i.e.

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \int_{\mathbb{R}} \left|\widehat{f}(t)\right|^2 \, \mathrm{d}t$$

for all $f \in \mathcal{S}(\mathbb{R})$.

(d) (opinion) The best normalized Gaussian is $g_\pi(x) = e^{-\pi x^2}$. This is because

$$\begin{aligned}\widehat{g_\pi}(t) &= \int_{\mathbb{R}} g_\pi(x) \cdot e^{-2\pi i t x}\,\mathrm{d}x = \int_{\mathbb{R}} e^{-\pi(x^2+2itx)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} e^{-\pi\left((x+it)^2+t^2\right)}\,\mathrm{d}x = g_\pi(t) \cdot \int_{\mathbb{R}} g_\pi(x+it)\,\mathrm{d}x \\ &= g_\pi(t) \cdot \int_{\mathbb{R}+it} g_\pi(z)\,\mathrm{d}x = g_\pi(t).\end{aligned}$$

So g_π is a fixed point of $\widehat{(-)}$.

Theorem (Poisson Summation)**.** *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \widehat{f}(k).$$

Proof. Suppose $f \in \mathcal{S}(\mathbb{R})$. There exists some $c > 0$ such that

$$|f(x)| \leq \frac{c}{1+x^2}$$

for all $x \in \mathbb{R}$. So for every $x \in [0,1]$ and every $n \in \mathbb{Z}$, we have

$$|f(x+n)| \leq \begin{cases} c & n = 0 \\ \dfrac{c}{1+(|n|-1)^2} & n \neq 0 \end{cases}$$

Then $\sum_{n\in\mathbb{Z}} f(x+n)$ converges absolutely and uniformly for all $x \in [0,1]$ by the Weierstrass M-test. The same works for $\sum_{n\in\mathbb{Z}} f'(x+n)$. Therefore $P_f(x) \coloneqq \sum_{n\in\mathbb{Z}} f(x+n)$ (the "periodization" of f) is continuous and integrable term-by-term for all $x \in [0,1]$, and differentiable term-by-term for all $x \in (0,1)$. But P_f is 1-periodic. So everything we said about $[0,1]$ extends to all of $\mathbb{R}$. Furthermore, everything we said about $(0,1)$ extends to $\mathbb{R} \setminus \mathbb{Z}$. Also note that P_f' is bounded where it exists, so by a theorem from last lecture, P_f is equal to its own (absolutely uniformly convergent) Fourier series:

$$P_f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}$$

where $c_k = \int_0^1 P_f(x) e^{-2\pi i k x}\,\mathrm{d}x$. So we may compute c_k as follows:

$$\begin{aligned}c_k &= \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i k x}\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y) e^{-2\pi i k (y-n)}\,\mathrm{d}y \\ &= \sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} f(y) e^{-2\pi i k y}\,\mathrm{d}y\right) e^{2\pi i k n} = \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y) e^{-2\pi i k y}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} f(y) e^{-2\pi i k y}\,\mathrm{d}y = \widehat{f}(k).\end{aligned}$$

Consequently:

$$\sum_{n\in\mathbb{Z}} f(x+n) = P_f(x) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$$

for all $x \in \mathbb{R}$. So evaluating this at 0 yields the proof. ■