MATH8510
Lecture 13 Notes

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Wrapping up on Fourier analysis

Last lecture, we showed that if $f \in L^1(\mathbb{R})$ and $\alpha \in \mathbb{R}^\times$, then

$$\widehat{f(\alpha x)}(t) = \frac{1}{|\alpha|}\widehat{f}\left(\frac{t}{\alpha}\right)$$

for all $t \in \mathbb{R}$. Furthermore, if $f \in \mathcal{S}(\mathbb{R})$, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \widehat{f}(k).$$

Corollary. *If $f \in \mathcal{S}(\mathbb{R})$ and $\alpha \in \mathbb{R}^\times$, then*

$$\sum_{n\in\mathbb{Z}} f(\alpha n) = \frac{1}{|\alpha|} \cdot \sum_{k\in\mathbb{Z}} \widehat{f}\left(\frac{k}{\alpha}\right)$$

Definition. The **Jacobi Theta Function** $\vartheta\colon (0,\infty) \to (0,\infty)$ is defined as

$$\vartheta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$$

Remark. Sometimes "the Jacobi Theta Function" refers to the holomorphic map

$$\vartheta(z;q) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i z}$$

where $q = e^{i\pi\tau}$ for some choice of $\tau \in \mathbb{C}$.

Proposition. $\vartheta(t) = \frac{1}{\sqrt{t}} \cdot \vartheta\left(\frac{1}{t}\right)$.

Proof. Recall $g_\pi(x) = e^{-\pi x^2}$. Then

$$\vartheta(t) = \sum_{n\in\mathbb{Z}} g_\pi(\sqrt{t}\cdot n) = \frac{1}{\sqrt{t}} \cdot \sum_{k\in\mathbb{Z}} \widehat{g_\pi}\left(\frac{k}{\sqrt{t}}\right) = \frac{1}{\sqrt{t}} \cdot \vartheta\left(\frac{1}{t}\right)$$

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Important Gadgets from Complex Analysis

$H(-)$ will denote the sheaf of holomorphic functions on $\mathbb{C}$, that is, the sheaf given by

$$H(\Omega) = \{f\colon \Omega \to \mathbb{C} \mid f' \text{ exists}\}.$$

Theorem. *If $f \in H(\Omega)$ and $\Omega' \subseteq \mathbb{C}$ is a connected open set containing Ω, then there is at most one $F \in H(\Omega')$ such that $F\big|_\Omega = f$. Such an F is called the **analytic continuation** of f.*

This is essentially the identity axiom for sheaves. In general, we want to know the largest possible Ω' for f. Even if the equation defining f on Ω is nonsense on Ω', if forced to remain holomorphic, the function "knows" how to extend itself to the largest possible domain.

Example. Let $\phi_+ = \frac{\sqrt{5}+1}{2} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ and $\phi_- = \frac{\sqrt{5}-1}{2} = \frac{1}{\phi_+}$, where $(F_n)_n = (0, 1, 1, 2, 3, 5, 8, 13, \ldots)$ is the Fibonacci sequence. Then $f(s) \coloneqq \sum_{n=0}^\infty F_n s^n$ defines a function $f \in H(\Omega)$ where

$$\Omega = \{s \in \mathbb{C} \mid |s| < \phi_-\}$$

Observe that $f(s)$ is undefined for $s \notin \overline{\Omega}$. Now let $\Omega' = \mathbb{C} \setminus \{\phi_+, \phi_-\}$, and note that $\Omega \subseteq \Omega'$ and Ω' is connected. Then the function $F \in H(\Omega')$ given by

$$F(s) = \frac{s}{(s-\phi_+)(s-\phi_-)}$$

satisfies $F\big|_\Omega = f$.

In summary, F is uniquely determined by the following properties:

1. $F(s) = \sum_{n=0}^\infty F_n s^n$ whenever $|s| < \phi_-$;
2. F is holomorphic;
3. The domain of F is as large as possible.

Definition. Let $\Omega = \{s \in \mathbb{C} \mid \Re(s) > 0\}$ and define $\gamma\colon \Omega \to \mathbb{C}$ by

$$\gamma(s) \coloneqq \int_0^\infty e^{-x} x^{s-1}\,\mathrm{d}x.$$

Then the **Gamma function** Γ is defined to be the analytic continuation of γ.

Exercise. Show that the following hold:

1. $\gamma(s+1) = s \cdot \gamma(s)$ for all $s \in \Omega$;
2. $\gamma \in H(\Omega)$;
3. $\gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Theorem (Bohr-Mollerup, 1922)**.** *γ is the unique function $f\colon (0,\infty) \to (0,\infty)$ satisfying*

1. $\log(f(x))$ *is convex;*

2. $f(1) = 1$*; and*

3. $f(x+1) = x \cdot f(x)$*.*

Definition. Let $\Omega' = \mathbb{C} \setminus \{0, -1, -2, -3, \ldots\}$ and $\Gamma \colon \Omega' \to \mathbb{C}$ by

$$\Gamma(s) := \begin{cases} \gamma(s) & \Re(s) > 0 \\ \frac{1}{s} \cdot \Gamma(s+1) & \Re(s) \leq 0 \end{cases}$$

Now we claim that Γ is holomorphic on Ω'. If $s \in \Omega' \setminus \Omega$, then $\Re(s) \leq 0$ and there exists a least $n \in \mathbb{N}$ such that $0 < \Re(s+n) \leq 1$. So

$$\begin{aligned} \Gamma(s) &= \frac{1}{s} \cdot \Gamma(s+1) = \frac{1}{s(s+1)} \cdot \Gamma(s+2) \\ &= \cdots = \frac{1}{s(s+1)(s+2)\cdots(s+(n-1))} \cdot \gamma(s+n) \end{aligned}$$

So we have the following immediate consequences:

1. $\gamma \in H(\Omega) \implies \Gamma \in H(\Omega')$, and

2. $\gamma \neq 0$ on Ω means Γ is never 0 on Ω', and thus Γ has a simple pole of residue $\frac{(-1)^n}{n!}$ at each $-n \in \{0, -1, -2, -3, \ldots\}$. In other words,

$$\lim_{s \to -n} (s+n)\Gamma(s) = \frac{(-1)^n}{n!}$$

In summary, Γ is the unique function $\Omega' \to \mathbb{C}$ with the following properties:

1. $\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} \, \mathrm{d}x$ if $\Re(s) > 0$;

2. $\Gamma \in H(\Omega')$; and

3. the domain of Γ cannot be made any larger.

So we will refer to Γ as *the* Gamma function.

Remark. We mentioned above that Γ is nonzero on its entire domain. This is very nice because, in particular, this means $\frac{1}{\Gamma} \in H(\mathbb{C})$ is an entire function. Furthermore, because Γ has a *simple* pole at each $-n \in \{0, -1, -2, -3, \ldots\}$, $\frac{1}{\Gamma}$ has a simple zero at each such $-n$.