MATH8510
Lecture 14 Notes

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September 23, 2022

Riemann's Functional Equation

Definition. Let $\Omega = \{s \in \mathbb{C} \mid \Re(s) > 1\}$, and define $\zeta\colon \Omega \to \mathbb{C}$ by

$$\zeta(s) \coloneqq \prod_{p\,\text{prime}} \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Our current goal is to find the analytic continuation of ζ to the largest possible $\Omega' \supseteq \Omega$, and give a formula for the analytic continuation when $s \in \Omega' \setminus \Omega$. We will start this process by defining an auxiliary function

$$f(s) \coloneqq \int_1^\infty \left(\frac{\vartheta(t) - 1}{2} \right) t^{\frac{s}{2} - 1} \, \mathrm{d}t.$$

Note that $f(s)$ is defined for all $s \in \mathbb{C}$:

1. If $\Re(s) \leq 2$, then $\left| t^{\frac{s}{2}} \right| \leq 1$ for all $t \in [0, \infty)$.
2. If $\Re(s) > 2$, then $\left| t^{\frac{s}{2} - 1} \right|$ increases with t, but not fast enough to prevent convergence, as $\frac{\vartheta(t) - 1}{2}$ decays so rapidly.

Exercise. Show that f is entire.

Definition. We define $\xi\colon \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$ by

$$\xi(s) \coloneqq f(s) + f(1 - s) - \left(\frac{1}{s} - \frac{1}{1 - s} \right).$$

By properties of f, we can see immediately that the following conditions hold:

1. $\xi \in H(\mathbb{C} \setminus \{0, 1\})$;
2. $\xi(1 - s) = \xi(s)$ for all $s \in \mathbb{C} \setminus \{0, 1\}$;
3. $\lim_{s \to 1} (s - 1) \cdot \xi(s) = 1$, and $\lim_{s \to 0} s \cdot \xi(s) = 1$.

But what really is ξ? Well,

$$\begin{aligned}
f(1-s) &= \int_1^\infty \left(\frac{\vartheta(u)-1}{2}\right) e^{\frac{1-s}{2}-1}\,\mathrm{d}u = \int_0^1 \left(\frac{\vartheta\left(\frac{1}{t}\right)-1}{2}\right) t^{-\frac{1-s}{2}-1}\,\mathrm{d}t \\
&= \int_0^1 \left(\frac{\sqrt{t}\vartheta(t)-\sqrt{t}}{2}+\frac{\sqrt{t}-1}{2}\right) t^{-\frac{(1-s)}{2}-1}\,\mathrm{d}t \\
&= \int_0^1 \left(\frac{\vartheta(t)-1}{2}\right) t^{\frac{s}{2}-1}\,\mathrm{d}t + \int_0^1 \frac{t^{\frac{s}{2}-1}}{2}\,\mathrm{d}t - \int_0^1 \frac{t^{\frac{s-1}{2}-1}}{2}\,\mathrm{d}t \\
&= \int_0^1 \left(\frac{\vartheta(t)-1}{2}\right) t^{\frac{s}{2}-1}\,\mathrm{d}t + \left(\frac{t^{\frac{s}{2}}}{s}\right)\Bigg|_0^1 - \left(\frac{t^{\frac{s-1}{2}}}{s-1}\right)\Bigg|_0^1 \\
&= \int_0^1 \left(\frac{\vartheta(t)-1}{2}\right) t^{\frac{s}{2}-1}\,\mathrm{d}t + \frac{1}{s} + \frac{1}{1-s}
\end{aligned}$$

So if $s \in \Omega$, then

$$\begin{aligned}
\xi(s) &= \int_0^\infty \left(\frac{\vartheta(t)-1}{2}\right) t^{\frac{s}{2}-1}\,\mathrm{d}t = \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right) t^{\frac{s}{2}}\,\mathrm{d}t \\
&\overset{\text{claim}}{\equiv} \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 t} t^{\frac{s}{2}-1}\,\mathrm{d}t = \pi^{-\frac{s}{2}} \sum_{n=1}^\infty \frac{1}{n^s} \int_0^\infty e^{-x} x^{\frac{s}{2}-1}\,\mathrm{d}x \\
&= \pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right) \zeta(s).
\end{aligned}$$

Thus, ξ is essentially the ζ function, along with some "front matter" $\pi^{-\frac{s}{2}} \cdot \Gamma\left(\frac{s}{2}\right)$.

Exercise. Verify the "claim" in the above computation.

Definition. Define $\mathcal{Z} : \mathbb{C} \setminus \{0,1\} \to \mathbb{C}$ by

$$\mathcal{Z}(s) \coloneqq \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \xi(s).$$

This is well-defined since $\frac{1}{\Gamma}$ is entire.

This function $\mathcal{Z}$ is holomorphic on its domain, and furthermore, the above computation shows that $\mathcal{Z}\big|_\Omega = \zeta$. So $\mathcal{Z}$ is *the* analytic continuation of ζ to $\mathbb{C} \setminus \{0,1\}$. We can also see that the symmetry $\xi(1-s) = \xi(s)$ shows us how to compute $\mathcal{Z}(s)$ for $\Re(s) < 0$:

$$\begin{aligned}
\mathcal{Z}(s) &= \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \xi(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \xi(1-s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \frac{\Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{1-s}{2}}} \cdot \mathcal{Z}(1-s) \\
&\overset{\text{claim}}{\equiv} \frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s)
\end{aligned}$$

And therefore, the following equation holds:

$$\mathcal{Z}(s) = \frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s). \tag{1}$$

This is called **Riemann's functional equation**.

Exercise. Verify the "claim" in the above computation.

Now we will note that the singularity of $\mathcal{Z}$ at $s = 0$ is removable, by the following quick computation:

$$\lim_{s\to 0} \mathcal{Z}(s) = \lim_{s\to 0} \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)\cdot \frac{s}{2}} = \frac{\xi(s)\cdot s}{2} = -\frac{1}{2}.$$

So at $s = 0$, we can define $\mathcal{Z}(0) := \lim_{s\to 0} \mathcal{Z}(s) = -\frac{1}{2}$. So by Riemann's Removable Singularity Theorem, $\mathcal{Z}'(0)$ also exists, and $\mathcal{Z}$ is well-defined and holomorphic on all of $\Omega' = \mathbb{C} \setminus \{1\}$.

Furthermore, the singularity of $\mathcal{Z}$ at $s = 1$ is a simple pole, and we can compute its residue:

$$\lim_{s\to 1} (s-1)\mathcal{Z}(s) = \lim_{s\to 1} \left(\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot (s-1)\,\xi(s) \right) = 1,$$

so $\mathcal{Z}$ has a simple pole of residue 1 at $s = 1$. In summary, $\mathcal{Z} : \mathbb{C} \setminus \{1\} \to \mathbb{C}$ is the unique function such that the following conditions hold:

1. $\mathcal{Z}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ whenever $\Re(s) > 1$;
2. $\mathcal{Z}$ is holomorphic on its domain;
3. The domain of $\mathcal{Z}$ cannot be made any bigger.

So $\mathcal{Z}$ will be the "true" ζ function, and we will write ζ to denote *this* function instead.

$$\zeta(s) = \begin{cases} \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} & \Re(s) > 1 \\ \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \xi(s) & 0 \le \Re(s) \le 1, s \ne 0, 1 \\ -\frac{1}{2} & s = 0 \\ \frac{(2\pi)^s}{\pi} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \prod_{p \text{ prime}} \frac{1}{1-p^{s-1}} & \Re(s) < 0 \end{cases}$$

If $k = 2m$ with $m \in \mathbb{N}$, then

$$\begin{aligned} \zeta(1-k) &= \frac{(2\pi)^{1-k}}{\pi} \cdot \sin\left(\frac{\pi(1-k)}{2}\right) \Gamma(k)\zeta(k) \\ &= \frac{2}{(2\pi)^m} \cdot (-1)^m (2m-1)! \cdot \frac{(2\pi)^m}{2(2m)!} (-1)^{m+1} B_{2m} \\ &= \frac{-B_{2m}}{2m} = -\frac{B_k}{k} \end{aligned}$$

If $k = 2m + 1$ with $m \in \mathbb{N}$, then

$$\zeta(1-k) = \frac{(\pi)^{1-k}}{\pi} \sin\left(\frac{\pi(1-k)}{2}\right) \Gamma(k)\zeta(k) = 0$$

So ζ has a "trivial zero" at $-2, -4, -6, -8, \ldots$

Corollary. $\zeta\left(1-k\right)=-\frac{B_k}{k}$ *for all integers* $k>1$. ■

Therefore, for any fixed prime number p, the function $\zeta_p \in H(\mathbb{C}\setminus\{1\})$ defined by

$$\zeta_p(s) \coloneqq \left(1-p^{-s}\right)\zeta(s)$$

satisfies the following properties:

i. $\zeta_p(s)=\prod_{q\neq p}\frac{1}{1-q^{-s}}$ for $\Re(s)>1$, and

ii. $\zeta_p(1-k)=-\left(1-p^{k-1}\right)\cdot\frac{B_k}{k}$ for all integers $k>1$.