MATH8510
Lecture 15 Notes

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p-adic Interpolation

Fix a prime p and $b \in \mathbb{N}$. Define $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by $f(s) = b^s$. Is it possible to extend f to $\mathbb{Z}_p$ in a *nice* way? Since $\mathbb{Z}_{\geq 0}$ is dense in $\mathbb{Z}_p$, the extension ought to be continuous. But we need to ensure $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is continuous with respect to the p-adic topology.

Case 1: if $p \mid b$, then $|b|_p < 1$ and $|f(s) - 1|_p = 1$, for all $s > 0$, by the strong triangle equality. Thus $\forall s_0 \in \mathbb{Z}_{\geq 0}$, we know that

$$|f(s) - f(s_0)|_p = |f(s_0)|_p \cdot |f(s - s_0) - 1|_p = |f(s_0)|_p > 0$$

for all $s > s_0$. So in this case, f fails to be continuous at *any* $s_0 \in \mathbb{Z}_{\geq 0}$.

Case 2: If $b \equiv 1 \pmod{p}$, then $b = 1 + kp$ for some $k \geq 0$. Thus for any $s_1 > s_2$ in $\mathbb{Z}_{\geq 0}$, we have

$$|f(s_1) - f(s_2)|_p = |b|_p^{s_2} \cdot \left|b^{s_1 - s_2} - 1\right|_p = \left| \sum_{j=1}^{s_1 - s_2} \binom{s_1 - s_2}{j} (kp)^j \right| \leq |s_1 - s_2|_p$$

So f is *uniformly* continuous on $\mathbb{Z}_{\geq 0}$, so it extends to a continuous function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, defined by

$$f(x) \coloneqq \lim_{n \to \infty} f(s_n)$$

where $(s_n)_n$ is any sequence in $(\mathbb{Z}_{\geq 0}, |\cdot|_p)$ converging to x.

Case 3: If $b \not\equiv 0, 1 \pmod{p}$, we again have a problem: the sequence $(s_n)_n = (p^n)_n$ converges to 0 in $(\mathbb{Z}_{\geq 0}, |\cdot|_p)$, but

$$f(s_n) = b^{s_n} = b^{p^n} \equiv b^{p^{n-1}} \equiv b^{p^{n-2}} \equiv \cdots \equiv b \not\equiv 1 \pmod{p}$$

This means $|f(s_n) - 1|_p = 1$ for all n, so $(f(s_n))_n \not\to 1$. Hence, f fails to be continuous at 0. So what do we do?

We can resolve this issue by fixing $r \in \{0, 1, \dots, p-2\}$ and restricting to the domain $S_r \coloneqq \{r + (p-1)m \mid m \in \mathbb{Z}_{\geq 0}\}$. Indeed, if $s \in S_r$, we have that

$$f(s) = b^r \left(b^{p-1}\right)^m$$

and $m \mapsto (b^{p-1})^m$ extends to $\mathbb{Z}_p$ by Case 1, because $b^{p-1} \equiv 1 \pmod{p}$. Thus, each of the $p-1$ different restrictions $f\colon S_0 \to \mathbb{Z}_{\geq 0}, f\colon S_1 \to \mathbb{Z}_{\geq 0}, \dots, f\colon S_{p-2} \to \mathbb{Z}_{\geq 0}$ extends to a continuous function $f\colon \mathbb{Z}_p \to \mathbb{Z}_p$, because each S_n is still dense in $\mathbb{Z}_p$.

We will see something similar to Case 3 when we try to interpolate $k \mapsto \zeta_p(1-k) = -(1-p^{k-1})\frac{B_k}{k}$.

p-adic Distributions

Fix a prime p, and a compact open subset $X \subseteq \mathbb{Q}_p$. Recall that a function $f\colon X \to \mathbb{Q}_p$ is said to be **locally constant** if $\forall x \in X, \exists U \subseteq X$ open with $x \in U$ such that $f(u) = f(x)\, \forall u \in U$. Write $\mathrm{LC}(X, \mathbb{Q}_p)$ to denote the set of all locally constant functions $f\colon X \to \mathbb{Q}_p$.

Exercise (Easy)**.** Show that $\mathrm{LC}(X, \mathbb{Q}_p)$ is a linear subspace of the $\mathbb{Q}_p$-vector space of continuous functions $X \to \mathbb{Q}_p$.

Definition. A **p-adic distribution** on X is a linear map

$$\mu\colon\ \mathrm{LC}(X, \mathbb{Q}_p) \to \mathbb{Q}_p.$$

Mimicking notations from analysis on $\mathbb{R}$ and $\mathbb{C}$, we write

$$\int_X f \,\mathrm{d}\mu = \int_X f(x)\,\mathrm{d}\mu(x)$$

instead of $\mu(f)$. An important fact from HW2 yields that every $f \in \mathrm{LC}(X, \mathbb{Q}_p)$ has the form $f = \sum_{i=1}^{k} \alpha_i \chi_{U_i}$, where each $\alpha_i \in \mathbb{Q}_p$ and χ_{U_i} is the indicator function of $U_i \subseteq \mathbb{Q}_p$, where the collection $\{U_i\}_{i=1}^{k}$ is a *disjoint* collection of open balls in X. In particular, this means

$$\int_X f\,\mathrm{d}\mu = \sum_{i=1}^{k} \alpha_i \int_X \chi_{U_i}\,\mathrm{d}\mu$$

so μ is entirely determined by the values

$$\mu(\chi_U) = \int_X \chi_U\,\mathrm{d}\mu$$

for open balls $U \subseteq X$. For this reason, we will often write $\mu(U) = \mu(\chi_U)$ when U is an open ball in X.

Proposition. *Let $\mathcal{B}_X$ be the set of open balls contained in X. If $\mu\colon \mathcal{B}_X \to \mathbb{Q}_p$ is any function satisfying*

$$\mu(c + p^n\mathbb{Z}_p) = \sum_{d=0}^{p-1} \mu\left(c + dp^n + p^{n+1}\mathbb{Z}_p\right) \tag{1}$$

for all $c + p^n\mathbb{Z}_p \subseteq X$, then μ extends uniquely to a p-adic distribution on X.

Proof sketch. If $U \subseteq X$ is compact and open, by a HW2 exercise, U is a disjoint union of open balls in X. Then eq. (1) yields additivity of μ over disjoint balls, as well as linearity of μ on $\mathrm{LC}(X, \mathbb{Q}_p)$. ■

We will talk next lecture about the *Haar distribution*, given by

$$\mu_{\text{Haar}}\left(c + p^n \mathbb{Z}_p\right) = \frac{1}{p^n}$$