MATH8510
Lecture 16 Notes

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p-adic Distributions (continued)

Recall. A **distribution** on X is a linear functional $\mu\colon \mathrm{LC}(X, \mathbb{Q}_p) \to \mathbb{Q}_p$.

Often the most natural distribution is the *Haar distribution*, defined by

$$\mu_{\text{Haar}}\left(c + p^n\mathbb{Z}_p\right) := \frac{1}{p^n}$$

for all open balls $c + p^n\mathbb{Z}_p$. The key property that Haar distribution has is translation invariance: $\mu_{\text{Haar}}(a+U) = \mu_{\text{Haar}}(U)$. However, on the p-adic numbers, the problem with the Haar distribution is that as n increases, $c + p^n\mathbb{Z}_p$ shrinks, but this means that

$$\left|\mu_{\text{Haar}}(c + p^n\mathbb{Z}_p)\right|_p = p^n,$$

i.e. the Haar distribution sees smaller open balls as "larger" in the p-adic sense. We will formalize this later, and see that the Haar distribution does not give us a p-adic measure.

Another simple example is the *point-mass distribution* for some $\alpha \in X$, defined by

$$\mu_\alpha(U) = \begin{cases} 1 & \alpha \in U \\ 0 & \alpha \notin U \end{cases}$$

This can be extended to finitely many "point masses" by taking some finite $A = \{\alpha_1, \ldots, \alpha_n\} \subseteq X$ and defining $\mu_A := \sum_{i=1}^n \mu_{\alpha_i}$.

Perhaps the most important distributions we will talk about, at least for the purposes of this course, are the *Bernoulli distributions*:

$$\mu_{B_k}\colon \mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p.$$

For each $k \geq 0$, we define μ_{B_k} on open balls $c + p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p$ by

$$\mu_{B_k}\left(c + p^n\mathbb{Z}_p\right) := p^{n(k-1)} B_k\left(\frac{c}{p^n}\right)$$

where we assume $c \in \{0, \ldots, p-1\}$.

It is natural to wonder why we even bothered to place the $B_k\left(\frac{c}{p^n}\right)$ term in the measure. The answer to this question will follow from this next Lemma:

Lemma. *Let $k \geq 0$. $B_k(x)$ is the unique degree k monic polynomial satisfying*

$$B_k(mt) = m^{k-1} \sum_{d=0}^{m-1} B_k\left(t + \frac{d}{m}\right) \tag{1}$$

for all $m \in \mathbb{N}$ and all $t \in \mathbb{C}$.

Proof. If $|z| < 2\pi$ and $x \in \mathbb{C}$, then

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k(x) \cdot \frac{z^k}{k!} = \frac{ze^{zx}}{e^z - 1} &= \left(\sum_{m=0}^{\infty} B_m \cdot \frac{z^m}{m!}\right)\left(\sum_{\ell=0}^{\infty} x^\ell \cdot \frac{z^\ell}{\ell!}\right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{\substack{m+\ell=k \\ m,\ell \geq 0}} B_m \cdot \frac{z^m}{m!} \cdot x^\ell \cdot \frac{z^\ell}{\ell!}\right) \\
&= \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} B_{k-\ell} \cdot \frac{k! \cdot x^\ell}{(k-\ell)!\ell!}\right) \frac{z^k}{k!}
\end{aligned}$$

and therefore:

$$\begin{aligned}
B_k(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} B_{k-\ell} x^\ell &= \binom{k}{k} B_0 x^k + \cdots \\
&= x^k + \cdots
\end{aligned}$$

Therefore, $B_k(x)$ is monic of degree k. Moreover, if $m \in \mathbb{N}$, then for all $t \in \mathbb{C}$, and all $z \in \mathbb{C}$ with $|z| < \frac{2\pi}{m}$, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(m^{k-1} \cdot \sum_{d=0}^{m-1} B_k\left(t + \frac{d}{m}\right)\right) \frac{z^k}{k!} &= \frac{1}{m} \cdot \sum_{d=0}^{m-1} \left(\sum_{k=0}^{\infty} B_k\left(t + \frac{d}{m}\right) \frac{(mz)^k}{k!}\right) \\
&= \frac{1}{m} \cdot \sum_{d=0}^{m-1} \left(\frac{(mz)e^{\left(t+\frac{d}{m}\right)(mz)}}{e^{mz} - 1}\right) \\
&= \frac{ze^{(mt)z}}{e^{mz} - 1} \cdot \sum_{d=0}^{m-1} (e^z)^d \\
&= \frac{ze^{(mt)z}}{e^{mz} - 1} \cdot \frac{(e^z)^m - 1}{e^z - 1} \\
&= \frac{ze^{(mt)z}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(mt) \cdot \frac{z^k}{k!}
\end{aligned}$$

Therefore, $B_k(mt) = m^{k-1} \sum_{d=0}^{m-1} B_k\left(t + \frac{d}{m}\right)$ for all $m \in \mathbb{N}$ and all $t \in \mathbb{C}$.

Now suppose g is monic of degree k and satisfies eq. (1). Then for any $\ell \in \mathbb{Z}$,

$$\begin{aligned}
\int_{\ell}^{\ell+1} (B_k(t) - g(t))\, \mathrm{d}t &= \lim_{m \to \infty} \left(\frac{1}{m} \sum_{d=0}^{m-1} \left(B_k\left(\ell + \frac{d}{m}\right) - g\left(\ell + \frac{d}{m}\right)\right)\right) \\
&= \lim_{m \to \infty} \frac{B_k(m\ell) - g(m\ell)}{m^k} = 0,
\end{aligned}$$

so on every interval $[\ell, \ell+1]$, the intermediate value theorem yields that $B_k(t) = g(t)$ for some $t \in [\ell, \ell+1]$. So the polynomial $B_k - g$ has infinitely many zeroes on $\mathbb{R}$, and is therefore the zero polynomial. This argument shows uniqueness of B_k as a polynomial satisfying eq. (1). ■

Now suppose $c + p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p$ with $c \in \{0, \ldots, p-1\}$, and $k \geq 0$. Then

$$\begin{aligned}
\mu_{B_k}\left(c + p^n\mathbb{Z}_p\right) &= p^{n(k-1)} B_k\left(\frac{c}{p^n}\right) = p^{n(k-1)} B_k\left(p \cdot \frac{c}{p^{n+1}}\right) \\
&= p^{n(k-1)}\left(p^{k-1}\sum_{d=0}^{p-1} B_k\left(\frac{c}{p^{n+1} + \frac{d}{p}}\right)\right) \\
&= \sum_{d=0}^{p-1} p^{(n+1)(k-1)} B_k\left(\frac{c + dp^n}{p^{n+1}}\right) \\
&= \sum_{d=0}^{p-1} \mu_{B_k}\left((c + dp^n) + p^{n+1}\mathbb{Z}_p\right)
\end{aligned}$$

By the Proposition from the previous lecture, since μ_{B_k} extends uniquely to a distribution

$$\mu_{B_k}\colon\ \mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p.$$

Now, the Bernoulli distributions still have the same issue as the Haar measure, in that smaller balls are measured to be "larger" in the p-adic sense. However, the Bernoulli distributions will be salvageable, as we will be able to normalize them. We will continue our discussion on Bernoulli distributions next lecture.