MATH8510
Lecture 17 Notes

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Fixing the Bernoulli Distributions

Recall. The **Bernoulli Distributions** are defined by

$$\begin{aligned} \mu_{B_k} \colon \mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) &\to \mathbb{Q}_p \\ c + p^n\mathbb{Z}_p &\mapsto p^{n(k-1)} B_k\left(\frac{c}{p^n}\right) \end{aligned}$$

where $c \in \{0, \dots, p-1\}$ and $k \geq 0$.

The problem with these distributions μ_{B_k}, much the same as the problem with the Haar distribution, is that $|\mu_{B_k}(U)|_p$ grows larger as U gets smaller; indeed,

$$\begin{aligned} \mu_{B_0}(c + p^n\mathbb{Z}_p) &= 1 \\ \mu_{B_1}(c + p^n\mathbb{Z}_p) &= \frac{c}{p^n} - \frac{1}{2} \\ \mu_{B_2}(c + p^n\mathbb{Z}_p) &= \frac{c^2}{p^n} - c + \frac{1}{p^n} \end{aligned}$$

and so on. These get larger in the p-adic sense as n gets larger, i.e. as the ball gets smaller.

Definition. A p-adic distribution $\mu \colon \mathrm{LC}(X, \mathbb{Q}_p)$ is called a ***p*-adic measure** if there exists an $M \geq 0$ such that, for all $U \subseteq X$ compact open,

$$|\mu(U)|_p \leq M.$$

So we can quantify the issue with the Bernoulli distributions in this language, by saying that μ_{B_k} is not a p-adic measure. However, there are some steps we can take to "regularize" these distributions, and will do so in a way which doesn't work for the Haar distribution, thus more robustly motivating the study of these perhaps initially strange distributions.

Remark. We take a moment to say a few things about distributions in general.

1. The set of distributions $\mathrm{LC}(X, \mathbb{Q}_p) \to \mathbb{Q}_p$ forms a $\mathbb{Q}_p$-vector space, as the dual space of the $\mathbb{Q}_p$-vector space $\mathrm{LC}(X, \mathbb{Q}_p)$. As such, we will denote by the space of distributions $(\mathrm{LC}(X, \mathbb{Q}_p))^*$.

2. The set of measures $\mu \in (\mathrm{LC}(X, \mathbb{Q}_p))^*$ forms a linear subspace of $(\mathrm{LC}(X, \mathbb{Q}_p))^*$.

3. If μ is a distribution (measure) on $\mathbb{Z}_p$, and $\alpha \in \mathbb{Z}_p^\times$, then $\mu' \in (\mathrm{LC}(X, \mathbb{Q}_p))^*$ defined by

$$\mu'(U) \coloneqq \mu(\alpha U)$$

is also a distribution (measure).

Definition. Let $k \geq 0$ be an integer, and let $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ with $\alpha \neq 1$. We define the **regularized Bernoulli Distribution** as such:

$$\mu_{k,\alpha} \colon \mathrm{LC}(\mathbb{Z}_p, \mathbb{Q}_p) \to \mathbb{Q}_p$$
$$\mu_{k,\alpha}(U) = \mu_{B_k}(U) - \alpha^{-k} \mu_{B_k}(\alpha U)$$

Our task now is to show that for each $k > 0$, there exists some $M_k \geq 0$ such that

$$\left|\mu_{k,\alpha}(c + p^n \mathbb{Z}_p)\right|_p \leq M_k$$

for all $n \geq 0$, and $c \in \{0, \ldots, p-1\}$.

Last time, we saw that the k^{th} Bernoulli polynomial has the following expression:

$$B_k(X) = \sum_{\ell=0}^{k} \binom{k}{\ell} B_{k-\ell} X^\ell = X^k - \frac{k}{2} X^{k-1} + \cdots + k B_{k-1} X + B_k$$

Bernoulli numbers are rational, so there exists a least $D_k \in \mathbb{N}$ such that $D_k B_k(X) \in \mathbb{Z}[X]$.

Theorem. *Given $k > 0$ and $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ with $\alpha \neq 1$, the following are true:*

1. $\left|\mu_{k,\alpha}(c + p^n \mathbb{Z}_p) - k c^{k-1} \mu_{1,\alpha}(c + p^n \mathbb{Z}_p)\right|_p \leq \left|\frac{1}{D_k}\right|_p \cdot p^{-n}$;

2. $\left|\mu_{k,\alpha}(c + p^n \mathbb{Z}_p)\right|_p \leq \left|\frac{1}{D_k}\right|_p$

for all $n \geq 0$ and $c \in \{0, \ldots, p-1\}$.

Proof. Fix $n \geq 0$, $c \in \{0, \ldots, p-1\}$, and note that $|\alpha|_p = 1$. So if $\{q\}$ denotes the fractional part of a rational number, then:

$$\alpha(c + p^n \mathbb{Z}_p) = p^n \left(\frac{\alpha c}{p^n} + \alpha \mathbb{Z}_p \right) = p^n \left(\left\{ \frac{\alpha c}{p^n} \right\} + \left\lfloor \frac{\alpha c}{p^n} \right\rfloor + \mathbb{Z}_p \right)$$
$$= p^n \left(\left\{ \frac{\alpha c}{p^n} \right\} + \mathbb{Z}_p \right) = p^n \left\{ \frac{\alpha c}{p^n} \right\} + p^n \mathbb{Z}_p$$

So $c_\alpha = p^n \left\{ \frac{\alpha c}{p^n} \right\}$ is the unique representative in $\{0, \ldots, p^n - 1\}$ such that

$$\alpha(c + p^n \mathbb{Z}_p) = c_\alpha + p^n \mathbb{Z}_p.$$

Therefore:

$$\begin{aligned}
D_k\mu_{k,\alpha}(c+p^n\mathbb{Z}_p) &= D_k\big(\mu_{B_k}(c+p^n\mathbb{Z}_p) - \alpha^{-k}\mu_{B_k}(\alpha(c+p^n\mathbb{Z}_p))\big) \\
&= D_k\left(p^{n(k-1)}B_k\left(\frac{c}{p^n}\right) - \alpha^{-k}p^{n(k-1)}B_k\left(\frac{c_\alpha}{p^n}\right)\right) \\
&= D_k\cdot\sum_{\ell=0}^{k}\binom{k}{\ell}B_\ell\left(p^{n(k-1)}\left(\frac{c}{p^n}\right)^{k-\ell} - \alpha^{-k}p^{n(k-1)}\left(\frac{c_\alpha}{p^n}\right)^{k-\ell}\right) \\
&= \sum_{\ell=0}^{k}\binom{k}{\ell}D_kB_\ell\left(c^{k-\ell} - \alpha^{-\ell}\left(\frac{c_\alpha}{\alpha}\right)^{k-\ell}\right)p^{n(\ell-1)} \qquad (\dagger)
\end{aligned}$$

But observe now that

$$\frac{c_\alpha}{\alpha} = \frac{p^n}{\alpha}\left\{\frac{\alpha c}{p^n}\right\} = \frac{p^n}{\alpha}\left(\frac{\alpha c}{p^n} - \left\lfloor\frac{\alpha c}{p^n}\right\rfloor\right) = c - \frac{p^n}{\alpha}\left\lfloor\frac{\alpha c}{p^n}\right\rfloor$$

So the ℓ^{th} summand of eq. $(\dagger)$ above is given by the following, when $\ell = 0$:

$$\begin{aligned}
D_k\cdot\left(c^k - \left(c - \frac{p^n}{\alpha}\left\lfloor\frac{\alpha c}{p^n}\right\rfloor\right)^k\right)p^{-n} &= D_k\left(c^k - \sum_{j=0}^{k}\binom{k}{j}\left(-\frac{p^n}{\alpha}\left\lfloor\frac{\alpha c}{p^n}\right\rfloor\right)^j c^{k-j}\right)p^n \\
&= D_k\left(kc^{k-1}\cdot\frac{p^n}{\alpha}\left\lfloor\frac{\alpha c}{p^n}\right\rfloor + z_0p^{2n}\right)p^{-n} \\
&= D_kkc^{k-1}\cdot\frac{1}{\alpha}\left\lfloor\frac{\alpha c}{p^n}\right\rfloor + D_kz_0p^n,
\end{aligned}$$

where $z_0 \in \mathbb{Z}_p$ is a parameter which we will not need to keep track of. If $\ell = 1$, then we have

$$k\cdot D_k\cdot -\frac{1}{2}\cdot\big(c^{k-1} - \alpha^{-1}(c^{k-1} + z_1p^n)\big) = D_kkc^{k-1}\cdot\left(\frac{\alpha^{-1}-1}{2}\right) + D_k\frac{\alpha^{-1}k}{2}z_1p^n$$

And finally, if $\ell > 1$, then the term is divisible by p^n, so we obtain:

$$\begin{aligned}
D_k\mu_{k,\alpha}(c+p^n\mathbb{Z}_p) &\equiv \sum_{\ell=0}^{1}\binom{k}{\ell}D_kB_\ell\left(c^{k-1} - \alpha^{-\ell}\left(\frac{c_\alpha}{\alpha}\right)^{k-\ell}\right)p^{n(\ell-1)} \pmod{p^n\mathbb{Z}_p} \\
&\equiv D_kkc^{k-1}\left(\frac{1}{\alpha}\left\lfloor\frac{\alpha c}{p^n}\right\rfloor + \frac{\alpha^{-1}-1}{2}\right) \pmod{p^n\mathbb{Z}_p} \\
&= D_kkc^{k-1}\mu_{1,\alpha}(c+p^n\mathbb{Z}_p).
\end{aligned}$$

This proves part (1) of the theorem. We will finish the proof of the theorem in the next lecture. ■