MATH8510
Lecture 18 Notes

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October 5, 2022

Measure and Integration on the p-adics.

Corollary (of the Theorem from last lecture). *For each k, $\mu_{k,\alpha}$ is a p-adic measure.* ∎

We care about the above Corollary because measures give us Riemann integration: fix a compact open $X \subseteq \mathbb{Q}_p$ and a p-adic measure $\mu \in (\mathrm{LC}(X, \mathbb{Q}_p))^*$. Recall that any $f \in \mathrm{LC}(X, \mathbb{Q}_p)$ can be written in the form

$$f = \sum_{i=1}^{k} a_i \chi_{B_i}$$

where $\{B_1, \ldots, B_k\}$ is a disjoint collection of open balls $B_i \subseteq X$, and $a_1, \ldots, a_k \in \mathbb{Q}_p$, and that in this case we have

$$\int_X f(x)\, \mathrm{d}\mu(x) = \int_X f \, \mathrm{d}\mu = \sum_{i=1}^{k} a_i \mu(B_i).$$

Definition. Suppose μ is a measure on a compact open $X \subseteq \mathbb{Q}_p$. The **Riemann integral** of a *continuous* $f \colon X \to \mathbb{Q}_p$ against μ is defined by

$$\int_X f(x)\, \mathrm{d}\mu(x) \coloneqq \lim_{n\to\infty} S_{n,(x_{c,n})}$$

where the n^{th} **Riemann sum** $S_{n,(x_{c,n})}$ is defined by

$$S_{n,(x_{c,n})} \coloneqq \sum_{c+p^n\mathbb{Z}_p \subseteq X} f(x_{c,n}) \mu(c + p^n \mathbb{Z}_p)$$

with **sample points** $x_{c,n} \in c + p^n\mathbb{Z}_p$.

Theorem. *The integral $\int_X f(x)\, \mathrm{d}\mu(x) = \int_X f \, \mathrm{d}\mu$ is a well-defined p-adic number, i.e.*

$$\lim_{n\to\infty} S_{n,(x_{c,n})}$$

exists in $\mathbb{Q}_p$ and is independent of our choice of sample points $(x_{c,n})$.

Proof. Since μ is a measure, there exists some $M \geq 0$ such that

$$|\mu(c + p^n\mathbb{Z}_p)|_p \leq M$$

for all $c + p^n\mathbb{Z}_p \subseteq X$. Since X is compact and open, there exists some $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, we may write X as a *finite* disjoint union of open balls of the form $c + p^n\mathbb{Z}_p$. Now for each $n \geq n_0$ and for each $c + p^n\mathbb{Z}_p$ appearing in the partition

$$X = \bigsqcup_c c + p^n\mathbb{Z}_p,$$

pick a sample point $x_{c,n} \in c + p^n\mathbb{Z}_p$ and define the Riemann sum:

$$S_{n,(x_{c,n})} = \sum_{c+p^n\mathbb{Z}_p \subseteq X} f(x_{c,n})\mu(c + p^n\mathbb{Z}_p)$$

Fix $\varepsilon > 0$ and choose $N = N(\varepsilon) \geq n_0$ (i.e. N is dependent on the choice of ε) such that

$$|f(x) - f(y)|_p < \frac{\varepsilon}{M+1}$$

wherever $x, y \in X$ satisfy $|x - y|_p \leq p^{-N}$; this holds because f is continuous and X is compact, so f is *uniformly* continuous on X. Now if N is fixed and $n \geq N$, we have

$$\begin{aligned}
&\left|S_{n+1,(x_{c,n+1})} - S_{n,(x_{c,n})}\right|_p \\
&= \left|\sum_{c'+p^{n+1}\mathbb{Z}_p \subseteq X} f(x_{c',n+1})\mu\big(c' + p^{n+1}\mathbb{Z}_p\big) - \sum_{c+p^n\mathbb{Z}_p} f(x_{c,n})\mu(c + p^n\mathbb{Z}_p)\right|_p \\
&= \left|\sum_{c+p^n\mathbb{Z}_p \subseteq X} \left(\sum_{\substack{c'+p^n\mathbb{Z}_p \\ \subseteq c+p^n\mathbb{Z}_p}} \big(f(x_{c',n+1})\mu\big(c' + p^{n+1}\mathbb{Z}_p\big) - f(x_{c,n})\mu\big(c' + p^{n+1}\mathbb{Z}_p\big)\big)\right)\right|_p \\
&= \left|\sum_{c+p^n\mathbb{Z}_p \subseteq X} \left(\sum_{\substack{c'+p^n\mathbb{Z}_p \\ \subseteq c+p^n\mathbb{Z}_p}} (f(x_{c',n+1}) - f(x_{c,n}))\mu\big(c' + p^{n+1}\mathbb{Z}_p\big)\right)\right|_p \\
&\leq \max_{c'+p^{n+1}\mathbb{Z}_p \subseteq X} \left\{\frac{\varepsilon}{M+1} \cdot M\right\} < \varepsilon.
\end{aligned}$$

Therefore, $\left(S_{n,(x_{c,n})}\right)_{n \geq n_0}$ is Cauchy in $\mathbb{Q}_p$, so $\lim_{n\to\infty} S_{n,(x_{c,n})}$ exists in $\mathbb{Q}_p$. Now to show that this sum is independent of the choice of sample points. For each $n \geq n_0$, choose sample points $y_{c,n} \in c + p^n\mathbb{Z}_p$ (possibly different from $x_{c,n}$). Just as before, fix $\varepsilon > 0$ and choose $N = N(\varepsilon) \geq n_0$. Then for each $n \geq N$, we have

$$\begin{aligned}
\left|S_{n,(x_{c,n})} - S_{n,(y_{c,n})}\right|_p &= \left|\sum_{c+p^n\mathbb{Z}_p \subseteq X} (f(x_{c,n}) - f(y_{c,n}))\mu(c + p^n\mathbb{Z}_p)\right|_p \\
&\leq \max_{c+p^n\mathbb{Z}_p \subseteq X} \left\{\frac{\varepsilon}{M+1} \cdot M\right\} < \varepsilon.
\end{aligned}$$

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