MATH8510
Lecture 19 Notes

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Some General Facts about Riemann Integration

Suppose $X \subseteq \mathbb{Q}_p$ is compact open, and μ a measure on X.

1. The $\mathbb{Q}_p$-vector space $C(X, \mathbb{Q}_p)$ of continuous functions $X \to \mathbb{Q}_p$ has a "sup norm" defined by

$$\|f\|_p \coloneqq \sup_{x \in X} |f(x)|_p$$

   which makes $C(X, \mathbb{Q}_p)$ into a complete (ultra-)metric space. $\mathrm{LC}(X, \mathbb{Q}_p)$ is a dense linear subspace of $C(X, \mathbb{Q}_p)$.

2. The Riemann integral $\Big(C(X, \mathbb{Q}_p), \|\cdot\|_p\Big) \to \Big(\mathbb{Q}_p, |\cdot|_p\Big)$ which sends a continuous function f to $\int_X f \, \mathrm{d}\mu$ is a continuous $\mathbb{Q}_p$-linear functional that extends the original definition of $\mu\colon \mathrm{LC}(X, \mathbb{Q}_p) \to \mathbb{Q}_p$.

Back to Bernoulli Distributions

We recently showed that for each $k > 0$ and each $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ with $\alpha \neq 1$, the regularized Bernoulli distribution $\mu_{k,\alpha} \in (\mathrm{LC}(X, \mathbb{Q}_p))^*$ is a *measure*, satisfying

$$\left|\mu_{k,\alpha}(c + p^n\mathbb{Z}_p) - kc^{k-1}\mu_{1,\alpha}(c + p^n\mathbb{Z}_p)\right|_p \leq p^{\mathrm{ord}_p(D_k) - n}$$

Theorem. *If X is a compact open subset of $\mathbb{Z}_p$, $k > 0$, and $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ with $\alpha \neq 1$, then*

$$\int_X \mathrm{d}\mu_{k,\alpha}(x) = \int_X kx^{k-1} \, \mathrm{d}\mu_{1,\alpha}(x).$$

Proof. Fix k, α as given and choose $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, X can be written as

$$X = \bigsqcup_c c + p^n\mathbb{Z}_p$$

Then for any such $n \geq n_0$, we have:

$$\int_X \mathrm{d}\mu_{k,\alpha}(x) = \mu_{k,\alpha}(x) = \sum_{c+p^n\mathbb{Z}_p \subseteq X} \mu_{k,\alpha}(c + p^n\mathbb{Z}_p)$$
$$= \sum_{c+p^n\mathbb{Z}_p} \left(kc^{k-1}\mu_{1,\alpha}(c + p^n\mathbb{Z}_p)\right) + \frac{z_n}{D_k} \cdot p^n$$

for some $z_n \in \mathbb{Z}_p$. Taking $n \to \infty$ yields

$$\left|\frac{z_n}{D_k} \cdot p^n\right|_p \to 0$$

and hence,

$$\int_X \mathrm{d}\mu_{k,\alpha}(x) = \int_X kx^{k-1}\,\mathrm{d}\mu_{1,\alpha}(x). \qquad \blacksquare$$

While the above argument is fairly straightforward, things are particularly simple when $X = p^n\mathbb{Z}_p \subseteq \mathbb{Z}_p$:

$$\int_{p^n\mathbb{Z}_p} kx^{k-1}\,\mathrm{d}\mu_{1,\alpha} = \int_{p^n\mathbb{Z}_p} 1\,\mathrm{d}\mu_{k,\alpha}(x) = \mu_{k,\alpha}(p^n\mathbb{Z}_p)$$
$$= \mu_{B_k}(p^n\mathbb{Z}_p) - \alpha^{-k}\mu_{B_k}(\alpha \cdot p^n\mathbb{Z}_p)$$
$$= p^{n(k-1)} \cdot B_k\left(\frac{0}{p^n}\right) - \alpha^{-k}p^{n(k-1)}B_k\left(\frac{0}{p^n}\right)$$
$$= p^{n(k-1)}\left(1 - \alpha^{-k}\right)B_k.$$

This integral computation has a nice consequence: if instead we consider the circle $X = p^n\mathbb{Z}_p^\times = p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$ inside of $\mathbb{Z}_p$, we see that

$$\int_{p^n\mathbb{Z}_p^\times} kx^{k-1}\,\mathrm{d}\mu_{1,\alpha}(x) = \int_{p^n\mathbb{Z}_p^\times} 1\,\mathrm{d}\mu_{k,\alpha}(x) = \mu_{k,\alpha}(p^n\mathbb{Z}_p)$$
$$= \mu_{k,\alpha}(p^n\mathbb{Z}_p) - \mu_{k,\alpha}\left(p^{n+1}\mathbb{Z}_p\right)$$
$$= p^{n(k-1)}\left(1 - \alpha^{-k}\right)\left(1 - p^{k-1}\right)B_k$$
$$= p^{n(k-1)} \cdot k\left(\alpha^{-1} - 1\right) \cdot \left(1 - p^{k-1}\right)\left(\frac{-B_k}{k}\right)$$
$$= p^{n(k-1)} \cdot k\left(\alpha^{-1} - 1\right) \cdot \zeta_p(1 - k).$$

This means that the function $\zeta_p \in H(\mathbb{C} \setminus \{1\})$ (originally defined by $\zeta_p(s) \coloneqq (1 - p^{-s})\zeta(s)$) satisfies

$$\zeta_p(1 - k) = \left(1 - p^{k-1}\right)\left(\frac{-B_k}{k}\right) = \frac{1}{p^{n(k-1)}(\alpha^{-k} - 1)} \cdot \frac{1}{k} \cdot \mu_{k,\alpha}\left(p^n\mathbb{Z}_p^\times\right)$$
$$= \frac{1}{p^{n(k-1)}(\alpha^{-k} - 1)} \cdot \frac{1}{k}\int_{p^n\mathbb{Z}_p^\times} \mathrm{d}\mu_{k,\alpha}(x)$$
$$= \frac{1}{p^{n(k-1)}(\alpha^{-k} - 1)}\int_{p^n\mathbb{Z}_p^\times} x^{k-1}\,\mathrm{d}\mu_{1,\alpha}$$

for all $k \geq 1$, and *any* choice of $n \geq 0$ and $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ with $\alpha \neq 1$.

Corollary. *Given any* $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ *with* $\alpha \neq 1$, *we have*

$$\zeta_p(1-k) = \left(1 - p^k\right)\left(\frac{-B_k}{k}\right) = \frac{1}{\alpha^{-k} - 1} \int_{\mathbb{Z}_p^\times} x^{k-1}\, \mathrm{d}\mu_{1,\alpha}(x). \qquad \blacksquare$$

The integral in the statement of the above Corollary given by

$$\int_{\mathbb{Z}_p^\times} x^{k-1}\, \mathrm{d}\mu_{1,\alpha}(x) \tag{1}$$

is the *Mellin-Mazur transform* of $\chi_{\mathbb{Z}_p^\times}$. We will p-adically interpolate $\zeta_p(1-k)$. Note that its dependence on k is much more transparent than that of B_k.

Theorem. *Fix* p *an odd prime and define* $S_{s_0} := \{s_0 + m(p-1) \mid m \in \mathbb{Z}_{\geq 0}\}$ *for each* $s_0 \in \{0, 1, \ldots, p-2\}$. *The following are true:*

1. *If* $s_0 \neq 0$, *then* $\frac{B_k}{k} \in \mathbb{Z}_{(p)}$ *for all* $k \in S_{s_0}$.
2. *If* $s_0 \neq 0$, *then* $\left(1 - p^{k-1}\right)\frac{B_k}{k} \equiv \left(1 - p^{k'-1}\right)\frac{B_{k'}}{k'} \pmod{p^{n+1}}$ *for all* $k, k' \in S_{s_0}$ *satisfying* $k \equiv k' \pmod{p^n}$.
3. *If* $s_0 = 0$, *then* $pB_k \equiv -1 \pmod{p}$ *for all positive* $k \in S_{s_0}$.

We will prove this theorem next lecture.