MATH8510
Lecture 2 Notes

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Absolute Values

Definition. Let K be a field. An **absolute value** on K is a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that the following conditions hold:

1. $|x| = 0 \iff x = 0$ for all $x \in K$.
2. $|xy| = |x| \cdot |y|$ for all $x, y \in K$.
3. $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

Proposition. *Every K has an absolute value.*

Proof. Let K be any field and let $|\cdot|_0$ be the constant function $x \mapsto 0$. ∎

A few initial comments: conditions (1) and (2) show that $|1| = 1$. (1) and (3) together endow K with a metric $d(x, y) = |x - y|$, and the metric topology is generated by open balls $B_r(y) \coloneqq \{x \in K \mid |x - y| < r\}$.

Condition (2) on its own shows that $|x| > 0$ for every unit $x \in K^\times = K \setminus \{0\}$ whenever $|\cdot|$ is not the "trivial absolute value" from the above proposition. In particular, it shows that $|\cdot| : K^\times \to \mathbb{R}_{>0}$ is a group homomorphism. Its kernel contains the roots of unity (which we will denote $\mu(K) \subseteq K$) since

$$\begin{aligned} x \in \mu(K) &\implies x^m = 1 \\ &\implies |x|^m = 1 \\ &\implies |x^m| = 1. \end{aligned}$$

Definition. Let $|\cdot| : K \to \mathbb{R}_{\geq 0}$ be an absolute value on a field K. the **value group** of the pair $(K, |\cdot|)$ is the image of the group homomorphism $|\cdot| : K^\times \to \mathbb{R}_{>0}$. We will denote the value group by Z_K.

Definition. Suppose K is a field with absolute value $|\cdot|$. We say that $|\cdot|$ is **Archimedean** if Z_K is unbounded with respect to $|\cdot|$, that is, for every $r \in \mathbb{R}_{>0}$, there exists some $x \in Z_K$ such that $|x| > r$. Otherwise, we say that $|\cdot|$ is **non-Archimedean**.

Theorem. *Suppose K is a field, $|\cdot|$ an absolute value on K. The following are equivalent:*

1. *$|\cdot|$ is non-Archimedean;*
2. *$|z| \leq 1$ for all $z \in \mathbb{Z}_K$;*
3. *$|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$.*

Proof. We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) first.

Suppose $\neg$(2), i.e. there exists some $z \in Z_K$ such that $|z| > 1$. Then given any $r > 0$, there exists some $n \in \mathbb{N}$ such that $|z^n| = |z|^n > r$. This shows $\neg$(1). So (1) $\Rightarrow$ (2).

Now suppose $|z| \leq 1$ for all $z \in Z_K$. Now suppose $x, y \in K$. Then for all $n \in \mathbb{N}$,

$$|x+y|^n = |(x+y)^n| = \left|\sum_{j=0}^{n} \binom{n}{j} \cdot x^j y^{n-j}\right| \leq \sum_{j=0}^{n} \left|\binom{n}{j}\right| \cdot |x|^j\, |y|^{n-j}$$

$$\leq \sum_{j=0}^{n} \max\{|x|, |y|\}^n = (n+1) \cdot \max\{|x|, |y|\}^n$$

Thus $|x+y| \leq \sqrt[n]{n+1}\,(\max\{|x|, |y|\})$. Since $\sqrt[n]{n+1} \to 1$ as $n \to \infty$, it follows that $|x+y| \leq \max\{|x|, |y|\}$. So (2) $\Rightarrow$ (3).

To see that (3) $\Rightarrow$ (2), we use induction; if $|z| \leq 1$ for some $z \in Z_K$, then we may observe that $|z \pm 1| \leq \max\{|z|, |\pm 1|\} \leq 1$. The fact that (2) $\Rightarrow$ (1) is immediate. ∎

Theorem. *Let K be a field, $|\cdot|$ a non-Archimedean absolute value on K. Then the closed unit ball $\mathcal{O}_K := \{x \in K \mid |x| \leq 1\}$ is a subring of K. Furthermore, the open unit ball $\mathfrak{m}_K := \{x \in K \mid |x| < 1\}$ is the unique maximal ideal in $\mathcal{O}_K$.*

Proof. Observe that $\pm 1 \in \mathcal{O}_K$, and for any $x, y \in \mathcal{O}_K$,

$$|x+y| \leq \max\{|x|, |y|\} \leq 1$$

and thus $x + y \in \mathcal{O}_K$. Similarly, $|xy| = |x| \cdot |y| \leq 1$, so $xy \in \mathcal{O}_K$. So indeed $\mathcal{O}_K$ is a subring of K. The proof that $\mathfrak{m}_K$ is an ideal of $\mathcal{O}_K$ is similarly clear. So it is left to show that $\mathfrak{m}_K$ is a maximal, and is unique with this property.

Let $I \subseteq \mathcal{O}_K$ be any ideal such that $I \not\subseteq \mathfrak{m}_K$. Then $\exists x \in I \setminus \mathfrak{m}_K$. Since $x \in \mathcal{O}_K \setminus \mathfrak{m}_K$, we see that $|x| = 1$. Furthermore, observe that $\mathcal{O}_K \setminus \mathfrak{m}_K = \{x \in K \mid |x| = 1\}$, so $x^{-1} \in \mathcal{O}_K$, or in other words, x is invertible in $\mathcal{O}_K$. So I contains a unit, and hence $I = \mathcal{O}_K$. ∎

Corollary. *If $(K, |\cdot|)$ is a non-Archimedean field, then $\mathcal{O}_K$ is a local ring.* ∎

Corollary. *The group of units in $\mathcal{O}_K$ is*

$$\mathcal{O}_K^\times = \mathcal{O}_K \setminus \mathfrak{m}_K = \{x \in K \mid |x| = 1\},$$

which is the "unit circle" in K. ∎

The quotient $\kappa := \mathcal{O}_K / \mathfrak{m}_K$ is a field, called the **residue field** of $(K, |\cdot|)$.