MATH8510
Lecture 21 Notes

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We have a Corollary of the theorem which we proved last lecture:

Corollary. *For $k = 1$ or k an even positive integer, we have*

$$B_k + \sum_{\substack{p \text{ prime} \\ (p-1)|k}} \frac{1}{p} \in \mathbb{Z}.$$

Proof. An exercise from Koblitz's text implies that part (3) of the Theorem holds for $p = 2$ as well.So let q be a prime such that $(q-1) \mid k$. Then $qB_k \equiv -1 \pmod{q\mathbb{Z}_{(q)}}$. In other words,

$$B_k + \frac{1}{q} \in \mathbb{Z}_{(q)}.$$

But any prime $p \neq q$ satisfies $\frac{1}{p} \in \mathbb{Z}_{(q)}$. Thus

$$B_k + \sum_{(p-1)|k} \frac{1}{p} \in \mathbb{Z}_{(q)}.$$

Now if $(q-1) \nmid k$, then $\frac{B_k}{k} \in \mathbb{Z}_{(q)}$ by Theorem (1). In particular, $B_k \in \mathbb{Z}_{(q)}$, and hence

$$B_k + \sum_{\substack{p \text{ prime} \\ (p-1)|k}} \frac{1}{p} \in \mathbb{Z}_{(q)}$$

for all primes q, therefore

$$B_k + \sum_{\substack{p \text{ prime} \\ (p-1)|k}} \frac{1}{p} \in \bigcap_q \mathbb{Z}_{(q)} = \mathbb{Z}. \qquad \blacksquare$$

Theorem (2) says that for $p > 2$ and $s_0 \in \{1, 2, \ldots, p-2\}$, the function $k \mapsto \zeta_p(1-k) = \frac{1}{\alpha^{-k}-1} \int_{\mathbb{Z}_p^\times} x^{k-1} \, d\mu_{1,\alpha}(x)$ is uniformly continuous (w.r.t. the p-adic metric) on $S_{s_0} = \{s_0 + (p-1)m \mid m \geq 0\}$. Hence:

Theorem. *For each $p > 2$ and $s_0 \in \{1, 2, \dots, p-2\}$, the s_0-branch of the p-adic zeta function $\zeta_{p,s_0} \colon \mathbb{Z}_{\geq 0} \to \mathbb{Q}_p$ given by*

$$\zeta_{p,s_0}(m) \coloneqq \frac{1}{\alpha^{-(s_0+(p-1)m)} - 1} \cdot \int_{\mathbb{Z}_p^\times} x^{s_0+(p-1)m-1} \, \mathrm{d}\mu_{1,\alpha}(x)$$

is uniformly continuous, and hence uniquely extends to a function $\zeta_{p,s_0} \in C(\mathbb{Z}_p, \mathbb{Q}_p)$, and is independent of our choice of $\alpha \in \mathbb{Z} \setminus p\mathbb{Z}$ with $\alpha \neq 1$.

Proof. Exercise. ∎

Indeed, this can be extended to the $s_0 = 0$ branch, provided we exclude $m = 0$. This is consistent with the fact that $\zeta_{p,0}(m) = \zeta_p(1-(p-1)m) = \left(1 - p^{(p-1)m-1}\right)\zeta(1-(p-1)m)$ is undefined at $m = 0$.

A bit of algebraic number theory.

Definition. Given a field K with an absolute value $|\cdot|$ and a vector space V over K, we call a function $\|\cdot\| \colon V \to \mathbb{R}_{\geq 0}$ a **norm** if the following conditions hold:

1. $\forall v \in V, \|v\| = 0 \iff v = 0$;
2. $\forall v \in V, \forall \alpha \in K, \|\alpha v\| = |\alpha| \cdot \|v\|$;
3. $\forall u, v \in V, \|u+v\| \leq \|u\| + \|v\|$.

We say that the norm $\|\cdot\|$ is **non-Archimedean** if the third condition can be refined to the strong triangle inequality:

$$\|u+v\| \leq \max\{\|u\|, \|v\|\}.$$

Just as an absolute value defines a metric topology on a field, a norm defines a metric topology on a vector space via the induced metric

$$(u, v) \mapsto \|u - v\|.$$

Example. If $\dim_K(V) < \infty$ and $\{v_1, \dots, v_n\}$ is a basis for V over K, then the "r-norm" for $r \geq 1$ and the "sup-norm" are respectively defined as follows, for $v = \sum_{i=1}^n a_i v_i \in V$:

$$\|v\|_r \coloneqq (|a_1|^r + \cdots + |a_n|^r)^{1/r}$$
$$\|v\|_\infty \coloneqq \sup\{|a_1|, \dots, |a_n|\}$$

Proposition. *If $(K, |\cdot|)$ is non-Archimedean, then so is $(V, \|\cdot\|_\infty)$.* ∎

For this reason, it seems that $\|\cdot\|_\infty$ is a norm best suited to V over a non-Archimedean field K. There is some bad news: all of these examples depend critically on a choice of basis. In particular, we want the topology on V to be independent of the choice of basis. However, there is some good news.

Definition. Two norms $\|\cdot\|, \|\cdot\|'$ on a vector space V over $(K, |\cdot|)$ are said to be **equivalent** if they induce the same topology.

The good news is the statement of the following exercise.

Exercise. Two norms $\|\cdot\|, \|\cdot\|'$ on a vector space V over $(K, |\cdot|)$ are equivalent if and only if there exist $C, D > 0$ such that $\|v\| \leq C \cdot \|v\|'$ and $\|v\|' \leq D\|v\|$ for all $v \in V$.

Theorem. *If $(K, |\cdot|)$ is complete w.r.t. its absolute value and V is a finite dimensional vector space over K, then all norms on V are equivalent. Furthermore, for any such norm $\|\cdot\|$, the metric space $(V, \|\cdot\|)$ is complete.*

Proof sketch. Fix a basis $\{v_1, \ldots, v_n\}$ for V over K. Check that $(V, \|\cdot\|_\infty)$ is complete (straightforward). Now suppose that $\|\cdot\|$ is any other norm on V, and let $C = n \cdot \max_{1\leq i\leq n}\{\|v_i\|\}$. Then $C > 0$ and each $v = \sum_{i=1}^n a_i v_i \in V$ satisfies

$$\|v\| \leq \sum_{i=1}^{n} |a_i|\|v_i\| \leq n \cdot \max_{1\leq i\leq n}\{|a_i|\} \cdot \max_{1\leq i\leq n}\{\|v_i\|\} \leq C \cdot \|v\|_\infty.$$

The other direction is more involved, but can be found on pp.174-176 in Gouvêa's text. Then the following easy exercises will complete the proof. ■

Exercise. Verify that equivalence of norms is transitive.

Exercise. Check that if two norms are equivalent, and V is complete with respect to one of them, then it is complete with respect to the other as well.