# MATH8510
# Lecture 3 Notes

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## Absolute Values (continued)

*Recall.* An **ultrametric space** is a metric space $(X, d)$ such that the strong triangle inequality holds; in other words, $\forall x, y, z \in X$,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

In this case, $d$ is called an **ultrametric**.

**Theorem** (Consequences of the strong triangle inequality)**.** *Let $(K, |\cdot|)$ be a non-Archimedean field. The following are true:*

1. *(strong triangle equality) if $x, y \in K$ and $x \neq y$, then $|x + y| = \max\{|x|, |y|\}$.*
2. *The metric $d : K \times K \to \mathbb{R}_{\geq 0}$ given by $d(x, y) = |x - y|$ is an ultrametric.*
3. *If $x, y, z \in K$ satisfy $d(x, y) < d(z, y)$, then $d(x, y) = d(z, y)$.*
4. *If $r > 0$, $x \in K$, and $y \in B_r(x)$, then $B_r(y) = B_r(x)$.*
5. *If $B_r(x) \cap B_s(y) \neq \emptyset$, then one of $B_r(x) \subseteq B_s(y)$ or $B_s(y) \subseteq B_r(x)$ holds.*
6. *Every circle $C_r(y) = \{x \in K \mid |y - x| = r\}$ with $y \in K$ and $r > 0$ is a clopen set in $(K, |\cdot|)$.*
7. *<u>all</u> balls $B_r(x)$ are clopen (equivalently, $\mathbb{Q}_p$ is totally disconnected).*

*Proof.* For (1), suppose $x, y \in K$ with $|x| > |y|$ WLOG. Then

$$|x| = |x + y - y| \leq \max\{|x + y|, |y|\} \leq \max\{\max\{|x|, |y|\}, |y|\} = |x|$$

and thus $|x + y| = |x| = \max\{|x|, |y|\}$.

For (2), if $x, y, z \in K$, then checking the ultrametric inequality amounts to observing

$$|x - y| = |(x - z) + (z - y)| \leq \max\{|x - z|, |x - y|\}$$

which suffices. We can see immediately that (3) follows from (1) and (2).

For (4), let $r > 0$ and suppose $y \in B_r(x)$ for some $x \in K$. If $z \in B_r(y)$, then

$$|z - x| \leq \max\{|z - y|, |y - x|\} < r$$

so $z \in B_r(x)$. This shows $B_r(y) \subseteq B_r(x)$. The other containment follows symmetrically.

For (5), let $x, y \in K$ and $r \geq s > 0$. If $B_r(x) \cap B_s(y) \neq \emptyset$, then since there is some $z_0 \in B_r(x) \cap B_s(y)$, $B_r(x) = B_r(z_0) \supseteq B_s(z_0) = B_s(y)$, by (4).

For (6), let $y \in K, r > 0$, and $x \in C_r(y)$. If $z \in B_r(x)$, then $|z - x| < r = |x - y|$, so by (1), we can see that $|z - y| = \max\{|z - x|, |x - y|\} = |x - y| = r$. So $B_r(x) \subseteq C_r(y)$. So we have shown that every point in $C_r(y)$ has an open neighbourhood contained therein, hence $C_r(y)$ is open. Hence $K \setminus C_r(y) = B_r(y) \cup \bigcup_{r'>r} C_{r'}(y)$ is open, so $C_r(y)$ is closed.

And finally, (7) follows from (6). ∎

**Theorem.** *If $|\cdot|$ and $|\cdot|'$ are absolute values on a field $K$, then the following are equivalent:*

a. *$|\cdot|$ and $|\cdot|'$ generate the same topology on $K$;*

b. *For all $x \in K$, we have $|x| < 1$ if and only if $|x|' < 1$.*

c. *There exists some $\alpha > 0$ such that $|x|^\alpha = |x|'$ for all $x \in K$.*

**Definition.** If $|\cdot|$ and $|\cdot|'$ are two absolute values on a field $K$ such that any of the conditions from the above theorem hold, we say that $|\cdot|$ and $|\cdot|'$ are **equivalent**, and write $|\cdot| \sim |\cdot|'$.

Before we prove this theorem, we will need a preliminary lemma.

**Lemma.** *Let $K$ be a field with absolute value $|\cdot|$. Any $x \in K$ satisfies $|x| < 1$ if and only if, for every open $U \subseteq K$ containing $0$, $U$ contains all but finitely many powers $\{x, x^2, x^3, \ldots\}$.*

*Proof.* Suppose $|x| < 1$. If $U \subseteq K$ is an open neighbourhood of 0, then since $|x^n - 0| = |x|^n \to 0$ as $n \to \infty$, there exists some $N \in \mathbb{N}$ such that $x^n \in U$ for all $n \geq N$.

For the other direction, we prove the contrapositive. Suppose that $|x| \geq 1$, so that $|x^n - 0| = |x|^n \geq |x| \geq 1$. Then $U := B_1(0)$ contains none of the $x^n$ terms. ∎

With that out of the way, we can prove the theorem.

*Proof of the Theorem.* We first show $(a) \Rightarrow (b)$. Suppose $U$ is open in $K_1 := (K, |\cdot|)$ if and only if it is open in $K_2 := (K, |\cdot|')$. Then $U$ is an open neighbourhood of 0 in $K_1$ if and only if it is an open neighbourhood of 0 in $K_2$. Then by the above Lemma, we find that $|x| < 1 \iff |x|' < 1$ for all $x \in K$.

Now we show $(b) \Rightarrow (c)$. Suppose $|x| < 1 \iff |x|' < 1$ for all $x \in K$. If $|\cdot| = |\cdot|_0$ (where $|\cdot|_0$ is the trivial absolute value $x \mapsto 0$), then necessarily $|\cdot|' = |\cdot|_0$ as well. Otherwise, $|\cdot|$ is nontrivial, so there exists some $x_0 \in K$ such that $|x_0| > 1$ (take some element with nonzero norm, and either it or its inverse will have norm $> 1$). Then $|x_0|' > 1$ as well. Let $\alpha := \frac{\log|x_0|'}{\log|x_0|}$ and note that $\alpha > 0$ and $|x_0|^\alpha = |x_0|'$.

Now suppose $x \in K$ satisfies $|x| > 1$ (and hence $|x|' > 1$ as well). We will show $|x|^\alpha = |x|'$ by contradiction. Suppose $|x|' < |x|^\alpha$. Then

$$\log|x|' < \alpha \log|x| = \log|x| \cdot \frac{\log|x_0|'}{\log|x_0|}$$

$$\implies \frac{\log|x|'}{\log|x_0|'} < \frac{\log|x|}{\log|x_0|}.$$

Then by density of $\mathbb{Q}$ we have some $m, n \in \mathbb{N}$ such that

$$\frac{\log |x|'}{\log |x_0|'} < \frac{m}{n} < \frac{\log |x|}{\log |x_0|}$$
$$\implies |x^n|' < |x_0^m|' \quad \text{and} \quad |x_0^m| < |x^n|$$

So $y = \frac{x^n}{x_0^m}$ satisfies $|y|' < 1$ and $|y| > 1$. This contradicts the assumption of $(b)$. Therefore $|x|^\alpha = |x|'$ for all $x \in K$.

As for $(c) \Rightarrow (a)$, we can just write down open balls and compare. ■