MATH8510
Lecture 4 Notes

Charlie Conneen

August 31, 2022

Absolute Values (continued)

Recall the following theorem from last lecture:

Theorem. *If $|\cdot|$ and $|\cdot|'$ are absolute values on a field K, then the following are equivalent:*

a. $|\cdot|$ and $|\cdot|'$ generate the same topology on K;

b. For all $x \in K$, we have $|x| < 1$ if and only if $|x|' < 1$.

c. There exists some $\alpha > 0$ such that $|x|^\alpha = |x|'$ for all $x \in K$.

If any of these equivalent conditions hold, we say that $|\cdot|$ and $|\cdot|'$ are **equivalent**, and write $|\cdot| \sim |\cdot|'$. So by the theorem, they generate the same topology if and only if their open unit balls are the same, if and only if one of $|\cdot|, |\cdot|'$ is a positive power of the other.

Archimedean Absolute Values on $\mathbb{Q}$.

Let $|\cdot|$ be an Archimedean absolute value on $\mathbb{Q}$. Recall that this means that the set

$$S = \{n \in \mathbb{N} \mid \ |n| > 1\}$$

is unbounded, and in particular, S is infinite.

Proposition. *Let $|\cdot|$ and S be as above. Then $\frac{\log|n|}{\log(n)}$ is independent of the choice of $n \in S$.*

Proof. Fix $m, n \in S$ and let $\ell \in \mathbb{N}$. Then there exists a unique base m expansion

$$n^\ell = a_0 + a_1 m + a_2 m^2 + \cdots + a_k m^k$$

with $a_i \in \{0, \dots, m-1\}$ for all i, and $a_k \neq 0$. In particular, $1 \leq m^k \leq n^\ell$, and thus

$$0 \leq k \leq \ell \cdot \frac{\log|n|}{\log(n)}$$

And therefore we obtain the following inequality:

$$\begin{aligned}
|n|^\ell = \left|n^\ell\right| = \sum_{i=0}^{k} |a_i| \cdot |m|^i &\le (k+1) \cdot m \cdot |m|^k \\
&\le \left(\ell \cdot \frac{\log(n)}{\log(m)} + 1\right) \cdot m \cdot \left(|m|^{\frac{\log(n)}{\log(m)}}\right)^\ell \\
\implies |n| &\le \sqrt[\ell]{\left(\ell \cdot \frac{\log(n)}{\log(m)} + 1\right) \cdot m} \cdot |m|^{\frac{\log(n)}{\log(m)}}
\end{aligned}$$

Since this holds for all $\ell \in \mathbb{N}$, taking $\ell \to \infty$ yields

$$|n| \le |m|^{\frac{\log(n)}{\log(m)}}$$

And therefore $\frac{\log|n|}{\log(n)} \le \frac{\log|m|}{\log(m)}$. By symmetry we can replace n and m with each other, so that this becomes an equality. Hence for all $n, m \in S$,

$$\frac{\log|n|}{\log(n)} = \frac{\log|m|}{\log(m)}$$

as required. ■

If $n \in S$, we will write $\alpha := \frac{\log|n|}{\log(n)}$ to denote the constant, which is independent of the choice of $n \in S$ by the above proposition.

Proposition. *Let α be as above. Then for all $m \in \mathbb{N}$, $\frac{\log|m|}{\log(m)} = \alpha$.*

Proof. Suppose $m \in \mathbb{N}$. Since $|m| > 0$ and S is unbounded, we can find some $n' \in S$ such that $|n'| > \frac{1}{|m|}$. This shows that $n'm \in S$ as well.

$$\begin{aligned}
\log|m| = \log|n'm| - \log|n'| &= \alpha \log(n'm) - \alpha \log(n') \\
&= \alpha \log(m).
\end{aligned}$$

Therefore $\frac{\log|m|}{\log(m)} = \alpha$. ■

Corollary. *$|m| = m^\alpha = |m|_\infty^\alpha$ for all $m \in \mathbb{N}$.* ■

Corollary. *$|x| = |x|_\infty^\alpha$ for all $x \in \mathbb{Q}$.* ■

Proposition. *If $\alpha > 1$, then $|\cdot| := |\cdot|_\infty^\alpha$ is not an absolute value.*

Proof. Consider $|1+1| = 2^\alpha > 2 = 1 + 1 = |1| + |1|$. ■

Theorem. *The set of all Archimedean absolute values on $\mathbb{Q}$ is precisely*

$$\{|\cdot|_\infty^\alpha \mid \alpha \in (0,1]\}.$$

Proof. Immediate consequence of the above statements. ■

Non-Archimedean Absolute Values on $\mathbb{Q}$.

Let $|\cdot|$ be a non-Archimedean absolute value on $\mathbb{Q}$. There is not much to say in the case that $|\cdot| = |\cdot|_0$, so we suppose that $|\cdot| \neq |\cdot|_0$ for the time being.

Since $|n| \leq 1$ for all $n \in \mathbb{N}$, there exists a least $n \in \mathbb{N}$ such that $|n| < 1$. Call this number p, and let $\alpha = -\frac{\log|p|}{\log(p)}$. Note that α is positive, since $\log(p)$ is positive, and $\log|p|$ is negative since $|p| < 1$.

Proposition. *Let $|\cdot|, p$, and α be as above. Then p is a prime, and $|n| = |n|_p^\alpha$ for all $n \in \mathbb{N}$.*

Proof. Let $a, b \in \mathbb{N}$ such that $ab = p$. Then $1 \leq a, b \leq p$, and

$$|a| \cdot |b| = |p| < 1,$$

so one of $|a| < 1$ or $|b| < 1$ must hold. By minimality of p, one of $a = p$ or $b = p$ must hold. This shows that p is prime.

Now suppose $n \in \mathbb{N}$, and write

$$n = q \cdot p + r$$

with $q \geq 0$ and $0 \leq r < p - 1$, by the division algorithm.

i. Suppose $p \nmid n$. Note that $0 \leq r < p - 1 \implies r < p$, so either $|r| = 0$ or $|r| = 1$. Since $p \nmid n$, it must be that $r \neq 0$, so $|r| = 1$. Furthermore, we have that $|qp| = |q| \cdot |p| < 1 = |r|$, so by the strong triangle inequality, we know that

$$|n| = |qp + r| = |r| = 1 = |n|_p^\alpha.$$

ii. Otherwise, suppose $p \mid n$. Then we may write

$$n = p^{\operatorname{ord}_p(n)} \cdot n'$$

where $p \nmid n'$, so by applying (i) to n', we have

$$|n| = |p|^{\operatorname{ord}_p(n)} \cdot |n'| = |p|^{-\alpha \operatorname{ord}_p(n)} = |n|_p^\alpha.$$

So our casework shows the claim. ∎

Corollary. *Every non-trivial non-Archimedean absolute value on $\mathbb{Q}$ is equivalent to a p-adic absolute value.* ∎

Corollary. *If $|\cdot|, p$, and α are as above, then $|x| = |x|_p^\alpha$ for all $x \in \mathbb{Q}$.* ∎

Remark. Fix a prime p. If p' is a different prime, then $x = \frac{p'}{p}$ satisfies $|x|_p > 1$ and $|x|_{p'} < 1$, so consequently $|\cdot|_{p'} \not\sim |\cdot|_p$. In other words, no two different primes will give similar absolute values. One can also check that $|\cdot|_p^\alpha$ is a non-Archimedean absolute value.

Corollary (Ostrowski's Theorem (1916))**.** *Every absolute value on $\mathbb{Q}$ is equivalent to precisely one of the following absolute values:*

$$|\cdot|_0, |\cdot|_2, |\cdot|_3, |\cdot|_5, |\cdot|_7, \ldots, |\cdot|_\infty$$ ∎