MATH8510
Lecture 5 Notes

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Completions

Definition. Let K be a field, $|\cdot|$ an absolute value on K. A **completion** of $(K, |\cdot|)$ is a field $\widehat{K}$ with an absolute value $\|\cdot\|$ along with the following data:

i. A field embedding $\iota : K \hookrightarrow \widehat{K}$ such that $\iota(K)$ is dense in $(\widehat{K}, \|\cdot\|)$;

ii. $\|\cdot\|$ extends $|\cdot|$, i.e. for all $x \in K$, $|x| = \|\iota(x)\|$.

iii. $(\widehat{K}, \|\cdot\|)$ is complete.

Theorem. *Given any field K with an absolute value $|\cdot|$, there exists a completion $(\widehat{K}, \|\cdot\|)$ which is unique up to (unique) isometric isomorphism.*

Proof. From a homework exercise, we know that the set $\mathcal{C}$ of Cauchy sequences in $(K, |\cdot|)$ is a ring with maximal ideal $\mathfrak{m}$ given by

$$\mathfrak{m} := \{(x_n)_n \in \mathcal{C} \mid \; |x_n| \to 0\}.$$

We define $\widehat{K} := \mathcal{C}/\mathfrak{m}$, which is a field since $\mathfrak{m}$ is a maximal ideal. To check the additional data, let $\iota : K \hookrightarrow \widehat{K}$ be given by $x \mapsto (x)_n + \mathfrak{m}$, i.e. every element of K is sent to the equivalence class of the corresponding constant sequence. This is indeed a field homomorphism, and is evidently injective.

To define $\|\cdot\| : \widehat{K} \to \mathbb{R}_{\geq 0}$, we let

$$\|(x_n)_n + \mathfrak{m}\| := \lim_{n\to\infty} |x_n| \,.$$

This is well-defined, since it is independent of the choice of representative sequence (easy). To see that this is an absolute value:

1. For any $\overline{x} = (x_n)_n + \mathfrak{m}$ and any $\overline{y} = (y_n)_n + \mathfrak{m}$,

$$\begin{aligned} \overline{x} \cdot \overline{y} &= (x_n \cdot y_n)_n + \mathfrak{m} \\ \implies \|\overline{x} \cdot \overline{y}\| &= \|x\| \cdot \|y\|. \end{aligned}$$

2. Similarly, we have

$$\|\overline{x}+\overline{y}\| \le \|\overline{x}\| + \|\overline{y}\|$$

in general, and

$$\|\overline{x}+\overline{y}\| \le \max\{\|\overline{x}\|, \|\overline{y}\|\}$$

if $(K, |\cdot|)$ is non-Archimedean.

We also know that $\|\iota(x)\| = |x|$ for all $x \in K$, because

$$\|\iota(x)\| = \|(x)_n + \mathfrak{m}\| = \lim_{n\to\infty} |x| = |x|.$$

We omit the proof of density of $\iota(K)$, as well as the proof of completeness of $(\widehat{K}, \|\cdot\|)$. To show that $(\widehat{K}, \|\cdot\|)$ is unique, if we have any other completion $(\widehat{K}', \|\cdot\|')$ equipped with an isometric embedding $\iota' : K \hookrightarrow \widehat{K}'$, then

$$\sigma \coloneqq \iota' \circ \iota^{-1} : \iota(K) \to \iota'(K)$$

is an isometric isomorphism. Then we use density and continuity to extend σ to an isometric isomorphism. ■

Remark. As it turns out, $\|\cdot\|$ is non-Archimedean if and only if $|\cdot|$ is non-Archimedean. In this case, $\|\widehat{K}^\times\| = |K^\times|$, or in other words, the value group does not grow larger under a non-Archimedean completion.

From now on, we refer to $\widehat{K}$ as *the* completion of $(K, |\cdot|)$, and we identify K with $\iota(K)$, and replace $\|\cdot\|$ with $|\cdot|$.

Definition. Fix a prime p. The **field of p-adic numbers**, written $\mathbb{Q}_p$, is the completion of $(\mathbb{Q}, |\cdot|_p)$.

This is unsatisfying: we have convinced ourselves that this object exists, and is unique up to isometric isomorphism, but this definition doesn't describe the p-adics any further than that. Furthermore, we are still left with the task of understanding how completions behave with respect to *equivalent* absolute values.

Absolute Values and Completions.

Let $|\cdot| \sim |\cdot|'$ be equivalent absolute values on a field K. If $\mathcal{C}$ and $\mathcal{C}'$ are the equivalence classes of Cauchy sequences in their respective absolute values, then we can quickly see that $\mathcal{C} = \mathcal{C}'$, and furthermore, their maximal ideals $\mathfrak{m} = \mathfrak{m}'$ agree. So $\mathcal{C}/\mathfrak{m} = \mathcal{C}'/\mathfrak{m}'$ as rings, and they have the same topology, since $|\cdot| \sim |\cdot|'$.

Theorem (Ostrowski's theorem, "completions" version)**.** *The full list of distinct completions of $\mathbb{Q}$ is*

$$\mathbb{Q} = \mathbb{Q}_0, \mathbb{Q}_2, \mathbb{Q}_3, \mathbb{Q}_5, \ldots, \mathbb{Q}_\infty = \mathbb{R}.$$ ■

Valuations

Definition. Let K be a field. A **valuation** on K is a function $v : K \to \mathbb{R} \cup \{\infty\}$ satisfying the following criteria:

i. $v(x) = \infty$ if and only if $x = 0$;

ii. $v(xy) = v(x) + v(y)$ for all $x, y \in K$.

iii. $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$.

Example. On $\mathbb{Q}$, ord_p is a valuation.

Proposition. *Given a valuation $v : K \to \mathbb{R} \cup \{\infty\}$, there exists a corresponding non-Archimedean absolute value on K, given by*

$$|\cdot| := e^{-v(\cdot)}. \qquad \blacksquare$$

Remark. We will see in the future that ord_p being a *discrete* valuation (in fact $\text{ord}_p(\mathbb{Q}^\times) = \mathbb{Z}$) will be very useful.